MATH.APP.410 Matrix Analysis (autumn 2025) / Mattila
Final Exam 16.10.2025

Neither calculators nor own materials are allowed in the exam. You do not have to return this question paper. The solutions for the problems can be found later from the course's Moodle page.

1. (a) Consider the vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbb{C}^3$, where $\boldsymbol{u} = (2i, 0, -1)$, $\boldsymbol{v} = (-i, i, 1+i)$ and $\boldsymbol{w} = (1+i, -i, -1+2i)$. Find the vector $\boldsymbol{x}$ that satisfies the equation
$$\boldsymbol{u} - \boldsymbol{v} + i\boldsymbol{x} = 2i\boldsymbol{x} + \boldsymbol{w}. \qquad \text{(3p)}$$
   (b) Suppose that A is a square matrix that satisfies the equation $A^2 - 2A + 5I = O$, where I is the $n \times n$ identity matrix and O is the zero matrix of the same size. Show that $A^{-1} = \frac{1}{5}(2I - A)$. Then make use of this result and find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$. (3p)
   Hint for the proof: try multiplying the matrix A with its claimed to be inverse.

2. Consider the 4×5 matrix
$$A = \begin{bmatrix} 2 & 1 & 1 & 3 & 4 \\ 4 & 3 & 2 & 1 & 0 \\ 6 & 5 & 4 & 7 & 8 \\ 0 & -1 & -1 & -3 & -4 \end{bmatrix}.$$
   (a) Find the LU decomposition for the matrix A. (4p)
   (b) Based on the LU decomposition found in part (a), determine the values of $\operatorname{rank}(A)$, $\dim(\mathcal{N}(A))$, $\operatorname{rank}(A^T)$ and $\dim(\mathcal{N}(A^T))$. (2p)

3. (a) Let $\mathcal{S} = \left\{ \begin{bmatrix} 2a+b-c \\ -a+b-c \\ -b+c \\ c \end{bmatrix} \middle| a, b, c \in \mathbb{R} \right\} \subset \mathbb{R}^4$. Find a basis for the subspace $\mathcal{S}$ and if possible, find a matrix A such that $\mathcal{R}(A) = \mathcal{S}$. (3p)
   (b) Show that the eigenvalues of the matrix $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, where $a, b \in \mathbb{R}$, are $\lambda_1 = a + ib$ and $\lambda_2 = a - ib$. (3p)

4. (a) Suppose that $X_n \in \mathbb{C}^{m \times n}$ is a matrix whose column vectors are linearly independent. The orthogonal projector matrix P that projects onto $\mathcal{R}(X_n)$ can be calculated by using the formula $P = X_n(X_n^* X_n)^{-1} X_n^*$. Show that this matrix P satisfies the equations $P^2 = P$ and $P^* = P$. (3p)
   (b) Suppose that the singular value decomposition of a certain matrix A is
$$A = U\Lambda V^* = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & i \end{bmatrix}^*.$$
   Use this decomposition and find
   (i) an orthonormal basis for the subspace $\mathcal{N}(A)$ and $\dim(\mathcal{N}(A))$, (1p)
   (ii) an orthonormal basis for the subspace $\mathcal{R}(A)$ and $\operatorname{rank}(A)$, (1p)
   (iii) the matrix $\Lambda^\dagger$ that is needed when calculating the pseudoinverse of the matrix A by the formula $A^\dagger = V\Lambda^\dagger U^*$. (1p)