TETA-5231 Financial Engineering

Exam
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Juho Kanniainen

This is a closed-book exam, a calculator allowed. You can answer in English or in Finnish. Good luck!

Question 1. Explain the following concepts and terms:

a) Volatility smile (1 p)

b) OTC-markets (1 p)

c) Chooser option (1 p)

d) Calibration (1 p)

e) Interest rate swap (1 p)

f) Arbitrage (1 p)

Question 2.

a) All Black-Scholes assumptions hold. Assume no dividends. The stock price is 100 EUR. The instantaneous riskless interest rate (short rate) is 5% in annual terms. Consider a one year European call option struck at-the-money. If volatility is zero (i.e. $\sigma = 0$), what is the call worth? After valuing the call, please tell me how to hedge the call (assuming you sold it). (2 p)

b) Suppose that IBM is trading at $75 per share. Consider a derivative security that pays exactly one dollar when IBM hits $100 for the first time. Show that by no-arbitrage, the derivative security cannot sell for more than $0.75. Ignore IBM's dividends, assume a riskless interest rate of zero, assume all assets are infinitely divisible, and ignore any taxes or transactions costs. (2 p)

c) At time t, let $C(t,T)$ be the price of a European call option on a non-dividend-paying stock, $S(t)$, with expiry $T > t$ and strike K with a payoff

$$C(T,T) = (S(T) - K)^+ .$$

Show that the no-arbitrage condition implies for all $t < T$ that

$$C(t.T) > (S(t) - KD(t,T))^+ ,$$

where the discount factor $D(t,T)$ equals to the price of a risk-free T-maturity zero-coupon bond at time t, $Z(t,T)$. (2 p)

Question 3.

a) Why do you get a smile/skew effect when you plot implied volatilities of options against their strike prices and maturity time? What can you say about it? Answer carefully. (3 p)

b) Suppose I don't know any mathematics. Explain to me why you use the riskless rate instead of the required return on the stock in the Black-Sholes formula. (3 p)

Question 4.

a) Consider a modified up-and-out call option that ceases to exist if the asset price reaches a barrier $H > S_0$, where S_0 is the current stock price. If $t_i < H$, for all $i = 1 \ldots n$, where $t_1, t_2, \ldots, t_n$ are monitoring dates, then an option pays

$$\max(S_T - \bar{S}, 0),$$

at maturity time T. Here $\bar{S}$ is the arithmetic average of daily stock prices, $\bar{S} = \frac{1}{n}\sum_{i=1}^{n} S_{t_i}$. In addition, $t_0 < t_1$ and $T \geq t_n$. This can been as an "Asian average strike up-and-out call".

Suppose that the risk-free interest rate, r, is a constant, and that the stock price follows geometric Brownian motion with a constant volatility of σ.

Please give a pseudo code that prices the above contract using *antithetic* variates. (6 p)

b) **EXTRA problem.** Give a pseudo code that prices the above contract under stochastic volatility using Heston (1993) model or some GARCH model. (2 p)

Please give two separate codes: (a) The code with geometric Brownian motion using antithetic variates and (b) the code under with stochastic volatility (the use of antithetic variates is not required in the second problem).

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Appendix

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- **Standard normal distribution**: A standard normal distribution is a normal distribution with zero mean and unit variance, given by the probability density function

$$n(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}.$$

- **Zero-coupon bond (discount bond)**: The value of a T-maturity zero-coupon bond at time t is

$$Z(t,T) = \exp\left(-\int_t^T r_s ds\right),$$

where r is a non-stochastic instantaneous interest rate (short rate). If r is stochastic, then

$$Z(t,T) = \mathbb{E}_t\left\{\exp\left(-\int_t^T r_s ds\right)\right\}.$$

The value of the zero-coupon bond is often denoted also by $P(t,T)$.

- **The binomial model**: Suppose that the stock is worth S_0 today and either US_0 or DS_0 after time Δt, where $U > 1 > D > 0$. The risk-neutral probability of an increase in stock price is expressed as

$$p = \frac{e^{r\Delta t} - D}{U - D}.$$

- **The dynamics of the risk-free asset**: Suppose that $r > 0$ is the instantaneous risk-free interest rate. Then the dynamics of an asset that earns rate r can be expressed as

$$dB(t) = rB(t)dt, \quad B(0) = B_0$$

where $B_0 > 0$.

- **Wiener process**: In continuous time, we write

$$dW(t) = \epsilon(t)\sqrt{dt},$$

and in discrete time

$$\Delta W(t) = \epsilon(t)\sqrt{\Delta t}.$$

- **The dynamics of a risky asset under geometric Brownian motion**: Suppose thet $\mu \in \mathbb{R}$ is the expected price appreciation and $\sigma > 0$ the instantaneous volatility, ϵ i.i.d. standard normal random variable, and $S_0 > 0$ the initial stock price. If the stock price is assumed to evolve according to geometric Brownian motion, we can write

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t). \quad S(0) = S_0. \tag{1}$$

- **Itô's lemma**: Suppose that stock price follows Eq. (1). If F is a function of stock price S, and time t, then $f(t) = F(t, S(t))$ satisfies the Itô equation

$$\begin{aligned} df(t) = &\left\{ F_t(t, S(t)) + \mu S(t) F_s(t, S(t)) + \frac{1}{2}\sigma^2 S(t)^2 F_{ss}(t, S(t)) \right\} dt \\ &+ \sigma S(t) F_s(t, S(t)) dW(t), \end{aligned}$$

where F_t, F_s, and F_{ss} denote $\frac{\partial F}{\partial t}, \frac{\partial F}{\partial s}$, and $\frac{\partial^2 F}{\partial S^2}$, respectively.

- **Analytical solution for the future stock price under geometric Brownian motion**: Suppose that $S_0 > 0$. Then

$$S(T) = S_0 \exp\left\{ \left(\mu - \frac{1}{2}\sigma^2 \right) T + \sigma W(T) \right\}.$$

- **Black-Scholes equation**: Suppose that $C(t, S(t))$ is the price of a European-type derivative asset written on stock $S(t)$. Then, with the assumptions of the Black-Scholes model, the price of the derivative asset satisfies the following differential equation whenever C is twice differentiable with respect to S and once with respect to t:

$$C_t + rSC_s + \frac{1}{2}\sigma^2 S^2 C_{ss} = rC.$$

- **Black-Scholes formula**: With the assumptions of the Black-Scholes model, the solution for a call option is

$$C(S(t), t) = S(t)N(d_1) - Ke^{-r(T-t)}N(d_2),$$

with $N(x)$ denoting cumulative standard normal distribution and

$$\begin{aligned} d_1(S(t), t) &= \frac{\ln S(t) - \ln K + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \\ d_2(S(t), t) &= \frac{\ln S(t) - \ln K + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}. \end{aligned}$$

- **Greeks:**

$$\Delta = \frac{\partial C}{\partial S} = e^{-q(T-t)}N(d_1),$$
$$\Gamma = \frac{\partial^2 C}{\partial S^2} = \frac{N'(d_1)e^{-q(T-t)}}{S\sigma\sqrt{T-t}}.$$
$$\Theta = \frac{\partial C}{\partial t} = -\frac{SN'(d_1)\sigma e^{-q(T-t)}}{2\sqrt{T-t}} + qSN(d_1)e^{-q(T-t)} - rKe^{-r(T-t)}N(d_2).$$
$$\mathcal{V} = \frac{\partial C}{\partial \sigma} = S\sqrt{T-t}N'(d_1)e^{-q(T-t)}.$$
$$\rho = \frac{\partial C}{\partial r} = K(T-t)e^{-r(T-t)}N(d_2)$$

- **Pricing under martingale measure**: Let X be as the numeraire (it can be a stock, the risk-free asset or any tradable asset). Moreover, $\Pi(t)$ is the price of a derivative security at time t that is priced according to the formula

$$\Pi(t) = X(t)\mathbb{E}_t^Q\left[\frac{\Pi(T)}{X(T)}\right],$$

where $T > t$ and $\mathbb{Q}$ is the martingale measure with X as the numeraire.

- **Girsanov's theorem**: Let $W(t)$ be a Brownian motion under probability measure $\mathbb{P}$. If

$$\int_0^t v(s)^2 ds < \infty$$

with probability one, then under equivalent probability measure $\mathbb{Q}$

$$W^Q(t) = W(t) - \int_0^t v(s)ds$$

is also a Brownian motion. The last equation can be rewritten as

$$dW(t) = dW^Q(t) + v(t)dt.$$

- **Currency derivatives**: Suppose that the spot exchange rate at time t is denoted by $X(t)$ and that the holder of the base currency earns the rate of r_b and the holder of the quote currency earns r_q and consider a call option which gives the owner of the option to buy one unit of the base currency (say euros) at the price K in the quote currency (say dollars). Then, with the Black-Scholes assumptions, the price of a call is given by

$$H(X(t),t) = X(t)e^{-r_b(T-t)}N(d_1) - Ke^{-r_q(T-t)}N(d_2),$$
$$d_1(X(t),t) = \frac{\ln X(t) - \ln K + \left(r_q - r_b + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$
$$d_2(X(t),t) = \frac{\ln X(t) - \ln K + \left(r_q - r_b - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}},$$

where σ is the instantaneous volatility of X.

- **Cash-or-nothing option**: The Black-Scholes price of a cash-or-nothing option is given by

$$C_{cn}(S(t),t) = e^{-r(T-t)}N(d_2),$$
$$d_2(S(t),t) = \frac{\ln S(t) - \ln K + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

- **Geometric average Asian option**: Suppose that Black-Sholes assumptions hold. Then the price of a geometric average price call is given by

$$C_{avp}^{geom}(S_0) = e^{-rT}\left(S_0 e^{\frac{1}{2}(r-\sigma^2/6)T}N(d_1)d - KN(d_2)\right),$$
$$d_1 = \frac{\ln S_0 - \ln K + \left(\frac{1}{2}\left(r - \frac{1}{6}\sigma^2\right) + \frac{1}{2}\sigma^2\right)T}{\frac{\sigma}{\sqrt{3}}\sqrt{T}},$$
$$d_2 = \frac{\ln S_0 - \ln K + \left(\frac{1}{2}\left(r - \frac{1}{6}\sigma^2\right) - \frac{1}{2}\sigma^2\right)T}{\frac{\sigma}{\sqrt{3}}\sqrt{T}}.$$

- **Two correlated random variables**: Suppose that the correlation between random variables ε_1 and ε_2 is ρ_{12}. Then we can write

$$\varepsilon_2 = \rho_{12}\varepsilon_1 + \sqrt{1-\rho_{12}^2}e_{12},$$

where e_{12} is $N(0,1)$ and independent of ε_1.

- **Forward rates**: Let $F(t;T_i,T_j)$ be the forward rate from time T_i to time T_j at time t, $T_j > T_i > t > 0$, and $P(t,T_j)$ the price of a T_j maturity zero-coupon bond (discount bond) at time t. Moreover, let τ_{ij} be a chosen time measure between T_i and T_j. Then the following relation holds:

$$F(t;T_i,T_j) = \frac{1}{\tau_{ij}}\left(\frac{P(t,T_i)}{P(t,T_j)} - 1\right).$$

- **The legs of interest rate swaps**: For simplicity, we denote $F_{i,i+1}(t)$ as $F_i(t)$, and $P(t,T_i)$ as $P_i(t)$. The present value of a cash flow on the fixed leg is

$$V_j^{\text{fix}} = K\tau_j P_{j+1}(t),$$

where K is the pre-agreed swap rate. The present value of a cashflow on the floating leg is

$$\begin{aligned} V_j^{\text{flo}} &= \tau_j L_j P_{j+1}(t) \\ &= \tau_j F_j(t) P_{j+1}(t) \\ &= P_j(t) - P_{j+1}(t). \end{aligned}$$

- **The market swap rate**: The market swap rate is a rate that makes the interest rate swap a fair contract at the present time, and it can be expressed as

$$X_{nN}(t) = \sum_{j=n}^{N-1} w_j(t) F_j(t),$$

where

$$w_j(t) = \frac{\tau_j P_{j+1}(t)}{\sum_{i=0}^{N-1} \tau_i P_{i+1}(t)}$$

or alternativelty as

$$X_{nN}(t) = \frac{P_n(t) - P_N(t)}{\sum_{j=n}^{N-1} \tau_j P_{j+1}(t)}.$$

- **The Black formula**: The Black-Scholes value of the caplet is

$$\begin{aligned}
\mathbf{CAP}_i(F_i(0)) &= \tau_i P_{i+1}(0) \mathbb{E}_0^{i+1}\left[(F_i(T_i) - K)^+\right] \\
&= \tau_i P_{i+1}(0) \left[F_i(0) N(d_1) - K N(d_2)\right], \\
d_1(F_i(0)) &= \frac{\ln(F_i(0)/K) + \frac{1}{2}\sigma_i^2 T_i}{\sigma_i \sqrt{T_i}}, \\
d_2(F_i(0)) &= \frac{\ln(F_i(0)/K) - \frac{1}{2}\sigma_i^2 T_i}{\sigma_i \sqrt{T_i}}.
\end{aligned}$$

- **Swaptions**: The Black-Sholes price of a swaption is given by

$$\begin{aligned}
\mathbf{PS}_{nN}(X_{nN}(0)) &= B_{nN}(0) \mathbb{E}_0^{nN}\left[X_{nN}(T_n) - K\right] \\
&= B_{nN}(0) \left[X_{nN}(0) N(d_1) - K N(d_2)\right], \\
d_1(X_{nN}(0)) &= \frac{\ln(X_{nN}(0)/K) + \frac{1}{2}\sigma_{nN}^2 T_n}{\sigma_{nN}\sqrt{T_n}}, \\
d_2(X_{nN}(0)) &= \frac{\ln(X_{nN}(0)/K) - \frac{1}{2}\sigma_{nN}^2 T_n}{\sigma_{nN}\sqrt{T_n}}.
\end{aligned}$$

- **Heston (1993) model**: The Heston model assumes that the stock price follows the diffusion

$$\begin{aligned}
dS(t) &= \mu S(t) dt + \sqrt{h(t)} S(t) dW_1(t), \\
dh(t) &= \kappa\left(\theta - h(t)\right) dt + \eta \sqrt{h(t)} dW_2(t),
\end{aligned}$$

with $S(0) = S_0 > 0$ and $h(0) = h_0 > 0$. Here μ is the expected price appriciation, h the squared instantaneous volatility (stochastic), $\kappa > 0$ the mean-reversion coefficient, θ the "long-run mean squared volatility", $\eta \in \mathbb{R}$ the "volatility of volatility", and W_1, W_2 Wiener processes with correlation $Corr[W_1, W_2] = \rho$.

- **Milstein discretization scheme for Heston model**:

$$h_{i+1} = h_i + \kappa(\theta - h_i)\Delta t + \eta\sqrt{h_i^+}\sqrt{\Delta t}\varepsilon_{2,i} + \frac{\eta^2}{4}\Delta t\left(\varepsilon_{2,i}^2 - 1\right).$$

- **GARCH -models**: The family of GARCH models can be generally expressed with the following pair of equations:

$$\ln\left(\frac{S_t}{S_{t-1}}\right) = r + \lambda h_t^{\psi} - \frac{1}{2}h_t + \sqrt{h_t}z_t,$$

$$h_t = \beta_0 + \sum_{i=1}^{p}\beta_i h_{t-i} + \sum_{i=1}^{q}\beta_{p+i} f\left(z_{t-i}\right),$$

where $r, \lambda > 0$, ψ is 0.5 or 1, $z_t \sim N(0,1), \mathbb{E}\left[z_{t_j} z_{t_k}\right] = 0$ for $j \neq k$, and $p, q \geq 1$. Here h_t denotes the conditional squared volatility.

- **GARCH(1,1) model**:

$$h_t = \beta_0 + \beta_1 h_{t-1} + \beta_2 z_{t-1}^2.$$

- **The most common specifications of the news impact function**:
  - Simple: $f(z_{t-1}) = z_{t-1}^2$
  - Leverage: ($\theta \geq 0$ is a constant)

$$f(z_{t-1}) = (z_{t-1} - \theta)^2.$$

  - News: ($\kappa, \theta \geq 0$ are constants)

$$f(z_{t-1}) = \{|z_{t-1} - \theta| - \kappa(z_{t-1} - \theta)\}^2.$$

  - Power: ($\theta, \gamma \geq 0$ are constants)

$$f(z_{t-1}) = (z_{t-1} - \theta)^{2\gamma}.$$

  - News&Power: ($\kappa, \theta, \gamma \geq 0$ are constants)

$$f(z_{t-1}) = \{|z_{t-1} - \theta| - \kappa(z_{t-1} - \theta)\}^{2\gamma}.$$

- **Leverage model under the risk-neutral measure**: Suppose that $\psi = 0.5$. Then the leverage model can be expressed under the risk-neutral measure as:

$$\begin{aligned} h_t &= \beta_0 + \beta_1 h_{t-1} + \beta_2 h_{t-1}\left(z_{t-1}^* - \lambda - \theta\right)^2 \\ &= \beta_0 + \beta_1 h_{t-1} + \beta_2 h_{t-1}\left(z_{t-1}^* - \theta^*\right)^2, \end{aligned}$$

where $\theta^* = \theta + \lambda$.

- **Log-likelihood function**:

$$l(r_1, \ldots, r_n) \propto \sum_{t=1}^{n}\left(-\ln\sigma_t^2 - \frac{r_t^2}{\sigma_t^2}\right).$$

Exponential function

Exponential function is defined by EXP(x) = e^x, where e is the constant 2.718... Note that EXP(-x) = 1/EXP(x). How to use:

- You need to find the $e^{2.51}$
- Find the row with x = 2.5
- Find the column 1, the number inside the cell is 12.3049, which is the result

x	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0	1	1.0101	1.0202	1.0305	1.0408	1.0513	1.0618	1.0725	1.0833	1.0942
0.1	1.1052	1.1163	1.1275	1.1388	1.1503	1.1618	1.1735	1.1853	1.1972	1.2093
0.2	1.2214	1.2337	1.2461	1.2586	1.2713	1.284	1.2969	1.31	1.3231	1.3364
0.3	1.3499	1.3634	1.3771	1.391	1.405	1.4191	1.4333	1.4477	1.4623	1.477
0.4	1.4918	1.5068	1.522	1.5373	1.5527	1.5683	1.5841	1.6	1.6161	1.6323
0.5	1.6487	1.6653	1.682	1.6989	1.716	1.7333	1.7507	1.7683	1.786	1.804
0.6	1.8221	1.8404	1.8589	1.8776	1.8965	1.9155	1.9348	1.9542	1.9739	1.9937
0.7	2.0138	2.034	2.0544	2.0751	2.0959	2.117	2.1383	2.1598	2.1815	2.2034
0.8	2.2255	2.2479	2.2705	2.2933	2.3164	2.3397	2.3632	2.3869	2.4109	2.4351
0.9	2.4596	2.4843	2.5093	2.5345	2.56	2.5857	2.6117	2.6379	2.6645	2.6912
N	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
1	2.7183	2.7456	2.7732	2.8011	2.8292	2.8577	2.8864	2.9154	2.9447	2.9743
1.1	3.0042	3.0344	3.0649	3.0957	3.1268	3.1582	3.1899	3.222	3.2544	3.2871
1.2	3.3201	3.3535	3.3872	3.4212	3.4556	3.4903	3.5254	3.5609	3.5966	3.6328
1.3	3.6693	3.7062	3.7434	3.781	3.819	3.8574	3.8962	3.9354	3.9749	4.0149
1.4	4.0552	4.096	4.1371	4.1787	4.2207	4.2631	4.306	4.3492	4.393	4.4371
1.5	4.4817	4.5267	4.5722	4.6182	4.6646	4.7115	4.7588	4.8067	4.855	4.9038
1.6	4.953	5.0028	5.0531	5.1039	5.1552	5.207	5.2593	5.3122	5.3656	5.4195
1.7	5.474	5.529	5.5845	5.6407	5.6973	5.7546	5.8124	5.8709	5.9299	5.9895
1.8	6.0497	6.1105	6.1719	6.2339	6.2965	6.3598	6.4237	6.4883	6.5535	6.6194
1.9	6.6859	6.7531	6.821	6.8895	6.9588	7.0287	7.0993	7.1707	7.2427	7.3155
N	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
2	7.3891	7.4633	7.5383	7.6141	7.6906	7.7679	7.846	7.9248	8.0045	8.0849
2.1	8.1662	8.2482	8.3311	8.4149	8.4994	8.5849	8.6711	8.7583	8.8463	8.9352
2.2	9.025	9.1157	9.2073	9.2999	9.3933	9.4877	9.5831	9.6794	9.7767	9.8749
2.3	9.9742	10.074	10.176	10.278	10.381	10.486	10.591	10.697	10.805	10.914
2.4	11.023	11.134	11.246	11.359	11.473	11.588	11.705	11.822	11.941	12.061
2.5	12.183	12.305	12.429	12.554	12.68	12.807	12.936	13.066	13.197	13.33
2.6	13.464	13.599	13.736	13.874	14.013	14.154	14.296	14.44	14.585	14.732
2.7	14.88	15.029	15.18	15.333	15.487	15.643	15.8	15.959	16.119	16.281
2.8	16.445	16.61	16.777	16.946	17.116	17.288	17.462	17.637	17.814	17.993
2.9	18.174	18.357	18.541	18.728	18.916	19.106	19.298	19.492	19.688	19.886
N	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
3	20.086	20.287	20.491	20.697	20.905	21.115	21.328	21.542	21.758	21.977
3.1	22.198	22.421	22.646	22.874	23.104	23.336	23.571	23.808	24.047	24.288
3.2	24.533	24.779	25.028	25.28	25.534	25.79	26.05	26.311	26.576	26.843
3.3	27.113	27.385	27.66	27.938	28.219	28.503	28.789	29.079	29.371	29.666
3.4	29.964	30.265	30.569	30.877	31.187	31.5	31.817	32.137	32.46	32.786
3.5	33.116	33.448	33.784	34.124	34.467	34.813	35.163	35.517	35.874	36.234
3.6	36.598	36.966	37.338	37.713	38.092	38.475	38.861	39.252	39.646	40.045
3.7	40.447	40.854	41.264	41.679	42.098	42.521	42.948	43.38	43.816	44.256
3.8	44.701	45.15	45.604	46.063	46.526	46.993	47.465	47.942	48.424	48.911
3.9	49.402	49.899	50.4	50.907	51.419	51.935	52.457	52.985	53.517	54.055
N	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
4	54.598	55.147	55.701	56.261	56.826	57.398	57.974	58.557	59.146	59.74
4.1	60.34	60.947	61.559	62.178	62.803	63.434	64.072	64.716	65.366	66.023
4.2	66.686	67.357	68.034	68.717	69.408	70.105	70.81	71.522	72.24	72.967
4.3	73.7	74.441	75.189	75.944	76.708	77.479	78.257	79.044	79.838	80.64
4.4	81.451	82.27	83.096	83.931	84.775	85.627	86.488	87.357	88.235	89.121
4.5	90.017	90.922	91.836	92.759	93.691	94.632	95.584	96.544	97.514	98.494
4.6	99.484	100.48	101.49	102.51	103.54	104.59	105.64	106.7	107.77	108.85
4.7	109.95	111.05	112.17	113.3	114.43	115.58	116.75	117.92	119.1	120.3
4.8	121.51	122.73	123.97	125.21	126.47	127.74	129.02	130.32	131.63	132.95
4.9	134.29	135.64	137	138.38	139.77	141.18	142.59	144.03	145.47	146.94