TTY / Talous ja rakentaminen

Rakennustekniikka

RAK-32300 ELEMENTTIMENETELMÄN PERUSTEET, 4 op

Syksy 2017

1. Välikoe ratkaisut

to 05.10.2017

1. Ratkaise Galerkinin menetelmällä differentiaaliyhtälö

$$\frac{d^2u}{dx^2}-\frac{du}{dx}=2, \qquad x\in[0,1], \qquad u(0)=u(1)=0$$

käyttäen kantafunktioita **$G_1(x)=x(x-1)$ ja $G_2(x)=x^2(x-1)$.** Vertaa saatua tulosta tarkkaan ratkaisuun, kun **x=0,5**.

$$u(x)_{Exact}=\left(2\cdot x+2\cdot e^x-2\cdot x\cdot e^1-2\right)\ /\ \left(e^1-1\right)$$

Basis functions $G_1(x)=x\cdot(x-1)$ and $G_2(x)=x^2\cdot(x-1)$

$\Rightarrow$

$$\tilde{u}(x)=Q_1\cdot x\cdot(x-1)+Q_2\cdot x^2\cdot(x-1)=Q_1\cdot(x^2-x)+Q_2\cdot(x^3-x^2)$$

$$\tilde{u}'(x)=Q_1\cdot(2x-1)+Q_2\cdot(3x^2-2x)$$

$$\tilde{u}''(x)=2Q_1+Q_2\cdot(6x-2)=2Q_1+2Q_2\cdot(3x-1)$$

$\Rightarrow$

$$L\tilde{u}-P=2Q_1+2Q_2\cdot(3x-1)-Q_1\cdot(2x-1)-Q_2\cdot(3x^2-2x)-2=3Q_1-2Q_2+8Q_2\cdot x-2Q_1\cdot x-3Q_2\cdot x^2-2$$

$$\phi=\phi_1\cdot(x^2-x)+\phi_2\cdot(x^3-x^2)$$

$\Rightarrow$

$$\int_0^1\phi\cdot(L\tilde{u}-P)\cdot dx=0 \qquad \forall\phi_i$$

$\Rightarrow$ $\qquad\qquad\downarrow$ Valitse ; Choose $\downarrow$

$$\begin{cases}\int_0^1 (x^2-x)\cdot\left(3Q_1-2Q_2+8Q_2\cdot x-2Q_1\cdot x-3Q_2\cdot x^2-2\right)\cdot dx=0 & \leftarrow(\phi_1=1,\quad \phi_2=0)\\ \int_0^1 (x^3-x^2)\cdot\left(3Q_1-2Q_2+8Q_2\cdot x-2Q_1\cdot x-3Q_2\cdot x^2-2\right)\cdot dx=0 & \leftarrow(\phi_1=0,\quad \phi_2=1)\end{cases}$$

$\Rightarrow$

$$\begin{cases}\int_0^1\left(3Q_1x^2-2Q_2x^2+8Q_2x^3-2Q_1x^3-3Q_2x^4-2x^2-3Q_1x+2Q_2x-8Q_2x^2+2Q_1x^2+3Q_2x^3+2x\right)\cdot dx=0\\ \int_0^1\left(3Q_1x^3-2Q_2x^3+8Q_2x^4-2Q_1x^4-3Q_2x^5-2x^3-3Q_1x^2+2Q_2x^2-8Q_2x^3+2Q_1x^3+3Q_2x^4+2x^2\right)\cdot dx=0\end{cases}$$

$\Rightarrow$

$$\begin{cases} \int_0^1 \left(3Q_1x^2 - 2Q_2x^2 + 8Q_2x^3 - 2Q_1x^3 - 3Q_2x^4 - 2x^2 - 3Q_1x + 2Q_2x - 8Q_2x^2 + 2Q_1x^2 + 3Q_2x^3 + 2x\right)\cdot dx = 0 \\ \int_0^1 \left(3Q_1x^3 - 2Q_2x^3 + 8Q_2x^4 - 2Q_1x^4 - 3Q_2x^5 - 2x^3 - 3Q_1x^2 + 2Q_2x^2 - 8Q_2x^3 + 2Q_1x^3 + 3Q_2x^4 + 2x^2\right)\cdot dx = 0 \end{cases}$$

$\Rightarrow$

$$\begin{cases} {}^1_0\!\!\left| \left(3Q_1\frac{x^3}{3} - 2Q_2\frac{x^3}{3} + 8Q_2\frac{x^4}{4} - 2Q_1\frac{x^4}{4} - 3Q_2\frac{x^5}{5} - 2\frac{x^3}{3} - 3Q_1\frac{x^2}{2} + 2Q_2\frac{x^2}{2} - 8Q_2\frac{x^3}{3} + 2Q_1\frac{x^3}{3} + 3Q_2\frac{x^4}{4} + 2\frac{x^2}{2}\right) = 0 \right. \\ {}^1_0\!\!\left| \left(3Q_1\frac{x^4}{4} - 2Q_2\frac{x^4}{4} + 8Q_2\frac{x^5}{5} - 2Q_1\frac{x^5}{5} - 3Q_2\frac{x^6}{6} - 2\frac{x^4}{4} - 3Q_1\frac{x^3}{3} + 2Q_2\frac{x^3}{3} - 8Q_2\frac{x^4}{4} + 2Q_1\frac{x^4}{4} + 3Q_2\frac{x^5}{5} + 2\frac{x^3}{3}\right) = 0 \right. \end{cases}$$

$\Rightarrow$

$$\begin{cases} \left(\frac{3}{3} - \frac{2}{4} - \frac{3}{2} + \frac{2}{3}\right)\cdot Q_1 + \left(-\frac{2}{3} + \frac{8}{4} - \frac{3}{5} + \frac{2}{2} - \frac{8}{3} + \frac{3}{4}\right)\cdot Q_2 = -\frac{1}{3} \\ \left(\frac{3}{4} - \frac{2}{5} - \frac{3}{3} + \frac{2}{4}\right)\cdot Q_1 + \left(-\frac{2}{4} + \frac{8}{5} - \frac{3}{6} + \frac{2}{3} - \frac{8}{4} + \frac{3}{5}\right)\cdot Q_2 = -\frac{1}{6} \end{cases} \Rightarrow \begin{bmatrix} -\frac{1}{3} & -\frac{11}{60} \\ -\frac{9}{60} & -\frac{8}{60} \end{bmatrix} \cdot \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \begin{Bmatrix} -\frac{1}{3} \\ -\frac{1}{6} \end{Bmatrix}$$

$\Rightarrow$

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \begin{bmatrix} -\frac{1}{3} & -\frac{11}{60} \\ -\frac{9}{60} & -\frac{8}{60} \end{bmatrix}^{-1} \cdot \begin{Bmatrix} -\frac{1}{3} \\ -\frac{1}{6} \end{Bmatrix} = \frac{1}{\left(\frac{8}{180} - \frac{9\cdot 11}{3600}\right)} \cdot \begin{bmatrix} -\frac{8}{60} & \frac{11}{60} \\ \frac{9}{60} & -\frac{1}{3} \end{bmatrix} \cdot \begin{Bmatrix} -\frac{1}{3} \\ -\frac{1}{6} \end{Bmatrix} = \begin{Bmatrix} \frac{50}{61} \\ \frac{20}{61} \end{Bmatrix} \approx \begin{Bmatrix} 0{,}82 \\ 0{,}33 \end{Bmatrix}$$

$\Rightarrow$

Solution

$$\tilde{u}(x) = \frac{50}{61}\cdot (x^2 - x) + \frac{20}{61}\cdot \left(x^3 - x^2\right)$$

$$\tilde{u}(0{,}5) = \frac{50}{61}\cdot (0{,}5^2 - 0{,}5) + \frac{20}{61}\cdot \left(0{,}5^3 - 0{,}5^2\right) \approx -0{,}2459 \qquad \leftarrow\leftarrow (1)$$

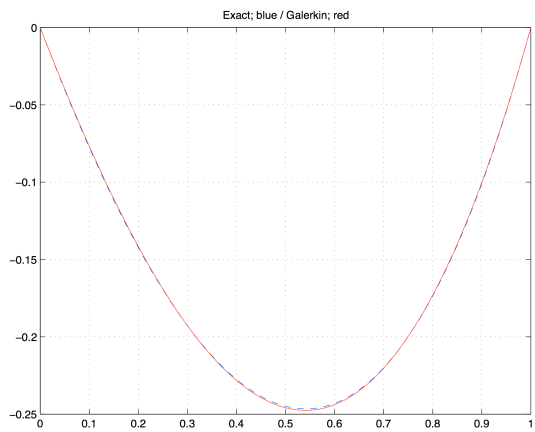
Exact solution

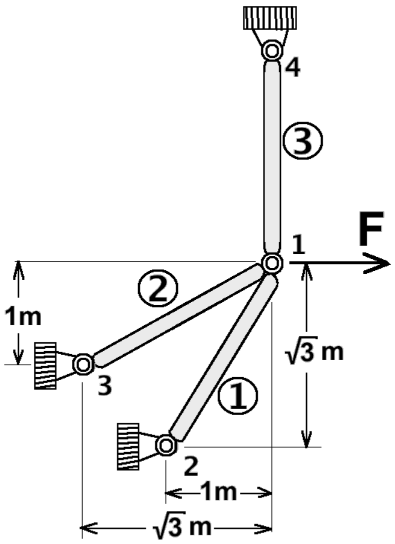
$$u(x)_{Exact} = \left(2\cdot x + 2\cdot e^x - 2\cdot x\cdot e^1 - 2\right) \,/\, \left(e^1 - 1\right)$$

$$u(0{,}5)_{Exact} = \left(2\cdot 0{,}5 + 2\cdot e^{0{,}5} - 2\cdot 0{,}5\cdot e^1 - 2\right) \,/\, \left(e^1 - 1\right) \approx -0{,}2449 \qquad \leftarrow\leftarrow (2)$$

$\Rightarrow \qquad (x = 0{,}5)$

The Galerkin approximate solution (1) $\cong$ The exact analytical solution of the differential equation (2)





2. Määritä kuvassa olevan kolmisauvaisen ristikon solmun **1** siirtymät ja sauvojen normaalijännitykset elementtimenetelmällä. Sauvat **1 ja 2** ovat pystysuoraan nähden kulmissa **30°** ja **60°** , vastaavasti. Sauva **3** on pystysuorassa. Sauvojen pituus on **2 m**. Materiaalin kimmomoduuli **E=200 GPa** ja sauvojen poikkipinta-ala **A=400 mm²** . Solmuun **1** vaikuttaa vaakasuora kuormitusvoima $\mathbf{F = 100000\ N}$. Laske lisäksi solmun **1** vaakasuuntainen siirtymä, jos tämän solmun pystysuuntainen siirtymä on estetty.

$$E = 200 \text{ GPa},\ A = 400mm^2, \quad F = 100000\ N, \quad L = 2\ m$$

Sauva 1 , Element 1

$$A_1 = 400mm^2 = 400\cdot 10^{-6} m^2, \quad 2 \to 1 \qquad l_1 = \frac{1}{2}, \quad m_1 = \frac{\sqrt{3}}{2}, \quad L_1 = 2\ m = L,$$

Sauva 2 , Element 2

$$A_2 = 400mm^2 = 400\cdot 10^{-6} m^2, \quad 3 \to 1 \qquad l_2 = \frac{\sqrt{3}}{2}, \quad m_2 = \frac{1}{2}, \quad L_2 = 2\ m = L,$$

Sauva 3 , Element 3

$$A_3 = 400mm^2 = 400\cdot 10^{-6} m^2, \quad 4 \to 1 \qquad l_3 = 0, \quad m_3 = -1, \quad L_3 = 2\ m = L,$$

Kaksi vapausastetta 2 DOF , (1 horizontal Q_1 , 2 vertical Q_2) , Node 1

$$k_e = \frac{EA}{l_e}\begin{bmatrix} l^2 & lm \\ lm & m^2 \end{bmatrix}, \quad k_1 = \frac{EA}{L}\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 1/4 & \sqrt{3}/4 \\ \sqrt{3}/4 & 3/4 \end{bmatrix} \end{array}\begin{array}{c} \\ 1 \\ 2 \end{array}, \quad k_2 = \frac{EA}{L}\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 3/4 & \sqrt{3}/4 \\ \sqrt{3}/4 & 1/4 \end{bmatrix} \end{array}\begin{array}{c} \\ 1 \\ 2 \end{array}, \quad k_3 = \frac{EA}{L}\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{array}\begin{array}{c} \\ 1 \\ 2 \end{array}$$

$\Rightarrow$

$$[K] = \sum_{i=1}^{3} k_i = \frac{EA}{L}\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 1 & \sqrt{3}/2 \\ \sqrt{3}/2 & 2 \end{bmatrix} \end{array}\begin{array}{c} \\ 1 \\ 2 \end{array} \qquad \{F\} = \begin{Bmatrix} 100000 \\ 0 \end{Bmatrix}\begin{array}{c} 1 \\ 2 \end{array}$$

$\Rightarrow$

$$[K]\{Q\} = \{F\} \qquad \det = 2 - \left(\sqrt{3}/2\right)^2 = \frac{5}{4} = 1{,}25$$

$\Rightarrow$

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = [K]^{-1}\{F\} = \frac{L}{EA}\cdot\frac{1}{\det}\begin{bmatrix} 2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1 \end{bmatrix}\begin{Bmatrix} 100000 \\ 0 \end{Bmatrix} =$$

$\to$

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \frac{2000}{200\cdot 10^3\cdot 400\cdot 1{,}25}\begin{Bmatrix} 200000 \\ -\sqrt{3}\cdot 50000 \end{Bmatrix} = \begin{Bmatrix} 4 \\ -\sqrt{3} \end{Bmatrix}\text{mm} \approx \begin{Bmatrix} 4{,}0 \\ -1{,}73 \end{Bmatrix}\text{mm}$$

Stress in the element 1

Node 2 fixed $\Rightarrow$ $q_1 = q_2 = 0$, $q_3 = Q_1$, $q_4 = Q_2$ *Node* 2 $\rightarrow$ Node 1

$$\sigma_1 = \frac{E}{L_1}\begin{bmatrix} -l_1 & -m_1 & l_1 & m_1 \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \frac{200\cdot 10^3}{2000}\left[\frac{1}{2}\cdot(4{,}0) + \frac{\sqrt{3}}{2}\cdot(-\sqrt{3})\right] = 50 \text{ MPa}$$

$\Rightarrow$

$\sigma_1 = 50$ MPa

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Stress in the element 2

Node 3 fixed $\Rightarrow$ $q_3 = q_4 = 0$, $q_1 = Q_1$, $q_2 = Q_2$ *Node* 3 $\rightarrow$ Node 1

$$\sigma_2 = \frac{E}{L_2}\begin{bmatrix} -l_2 & -m_2 & l_2 & m_2 \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \frac{200\cdot 10^3}{2000}\left[\frac{\sqrt{3}}{2}\cdot(4{,}0) + \frac{1}{2}\cdot(-\sqrt{3})\right] = \frac{300\cdot\sqrt{3}}{2} \approx 260 \text{ MPa}$$

$\Rightarrow$

$\sigma_2 = 260$ MPa

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Stress in the element 3

Node 4 fixed $\Rightarrow$ $q_3 = q_4 = 0$, $q_1 = Q_1$, $q_2 = Q_2$ *Node* 4 $\rightarrow$ Node 1

$$\sigma_3 = \frac{E}{L_3}\begin{bmatrix} -l_3 & -m_3 & l_3 & m_3 \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \frac{200\cdot 10^3}{2000}\left[0\cdot(4{,}0) - 1\cdot(-\sqrt{3})\right] = \sqrt{3}\cdot 100 \text{ MPa} \approx 173 \text{ MPa}$$

$\Rightarrow$

$\sigma_3 = 173$ MPa

Vain voiman suuntainen liike solmussa 1; Only horizontal displacement of the node 1

Vapausaste Q_2 eliminoidaan ; Elimination of Q_2

$$[K]=\sum_{i=1}^{3}k_i=\frac{EA}{L}\begin{array}{cc} \begin{matrix}1 & \phantom{\sqrt{3}/2}\end{matrix} & \\ \begin{bmatrix}1 & \sqrt{3}/2\\ \sqrt{3}/2 & 2\end{bmatrix} & \begin{matrix}1\\ \end{matrix}\end{array} \qquad \{F\}=\begin{array}{cc}\begin{Bmatrix}100000\\ 0\end{Bmatrix} & \begin{matrix}1\\ \end{matrix}\end{array}$$

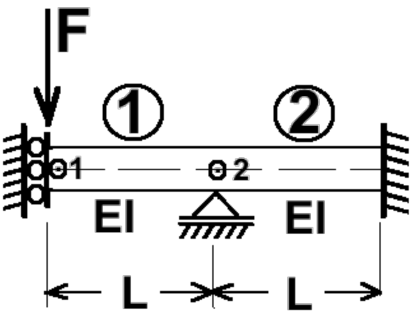
$\Rightarrow$

$$\frac{E\cdot A}{L}\cdot 1\cdot Q_3=100000$$

$\Rightarrow$

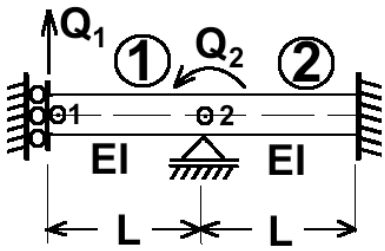
Siirtymä : Displacement Q_3

$$Q_3=\frac{L}{E\cdot A}\cdot 100000=\frac{2000}{200000\cdot 400}\cdot 100000=2{,}5\ \text{mm} \quad \leftarrow\leftarrow$$



3. Määritä elementtimenetelmällä kuvan päistään ja keskeltä tuetun vaakapalkin kiertymä ja siirtymä, kun solmussa **1** vaikuttaa pistevoima **F** alaspäin. Määritä lisäksi elementtien keskipisteiden taipumat ja taivutusmomentti liukutuen kohdalla solmussa **1**. Palkin taivutusjäykkyys on **EI**. Palkin pituus on **2L**. Palkki on venymätön. Käytä ratkaisussa kahta elementtiä.

E = 200 GPa, I = 10^{-4} m^4, L = 2 m, F = 15 kN



$$[k_1] = \frac{EI}{L^3}\begin{array}{c} \begin{array}{cccc} 1 & & 2 & \end{array} \\ \begin{bmatrix} \mathbf{12} & 6L & -12 & \mathbf{6L} \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ \mathbf{6L} & 2L^2 & -6L & \mathbf{4L^2} \end{bmatrix} \end{array}\begin{array}{c} 1 \\ \\ \\ 2 \end{array} \quad , \quad [k_2] = \frac{EI}{L^3}\begin{array}{c} \begin{array}{cccc} & 2 & & \end{array} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & \mathbf{4L^2} & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \end{array}\begin{array}{c} \\ 2 \\ \\ \\ \end{array}$$

$\Rightarrow$

After elimination

$$[\mathrm{K}] = \frac{EI}{L^3}\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 12 & 6L \\ 6L & 8L^2 \end{bmatrix} \end{array}\begin{array}{c} 1 \\ 2 \end{array} \quad , \quad \det = 12\cdot 8L^2 - (6L)\cdot(6L) = 60L^2$$

Loading vector Moment and Equivalent nodal forces (moment and force)

Node 1 Element 2 (Nodes 1 and 2)

$$\{\mathrm{F}\} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{Bmatrix} -F \\ 0 \end{Bmatrix} = \begin{Bmatrix} -15000 \\ 0 \end{Bmatrix}\mathrm{Nmm}$$

$\Rightarrow$

Siirtymä Q_1 ja kiertymä Q_2 Displacement Q_1 and slope Q_2

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = [K]^{-1}\cdot\{F\} = \frac{L^3}{EI}\cdot\frac{1}{\det}\cdot\begin{bmatrix} 8L^2 & -6L \\ -6L & 12 \end{bmatrix}\cdot\begin{Bmatrix} -F \\ 0 \end{Bmatrix} = \frac{L}{60EI}\cdot\begin{Bmatrix} -F\cdot 8L^2 \\ F\cdot 6L \end{Bmatrix}$$

$\Rightarrow$

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \frac{L}{60EI}\cdot\begin{Bmatrix} -F\cdot 8L^2 \\ F\cdot 6L \end{Bmatrix} = \frac{2000}{60\cdot 200000\cdot 10^8}\cdot\begin{Bmatrix} -15000\cdot 8\cdot 2000^2 \\ 15000\cdot 6\cdot 2000 \end{Bmatrix} \approx \begin{Bmatrix} -0{,}8\ \mathrm{mm} \\ 3{,}0\cdot 10^{-4} \end{Bmatrix} \quad \leftarrow$$

$\Rightarrow$

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} \approx \begin{Bmatrix} -0{,}8\ \text{mm} \\ 0{,}017° \end{Bmatrix}$$

Taipuma vasemmanpuoleisten tukien keskellä
The deflection at the midpoint between the left supports

Element 1

$$H_1 = \frac{1}{4}(2 - 3\xi + \xi^3) \quad , \quad H_2 = \frac{1}{4}(1 - \xi - \xi^2 + \xi^3)$$
$$H_3 = \frac{1}{4}(2 + 3\xi - \xi^3) \quad , \quad H_4 = \frac{1}{4}(-1 - \xi + \xi^2 + \xi^3)$$
$$v(\xi) = H_1 q_1 + \frac{l_e}{2} H_2 q_2 + H_3 q_3 + \frac{l_e}{2} H_4 q_4$$

$$q_1 = Q_1 \quad , \quad q_2 = q_3 = 0 \quad , \quad q_4 = Q_2$$

$\Rightarrow$

$$v(0) = H_1(0) \cdot q_1 + \frac{L}{2} \cdot H_4(0) \cdot Q_2 = -\frac{1}{2} \cdot \frac{8F \cdot L^3}{60EI} - \frac{L}{2} \cdot \frac{1}{4} \cdot \frac{6F \cdot L^2}{60EI} = -\frac{38FL^3}{480EI} \approx -0{,}475\ \text{mm} \qquad \leftarrow$$

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Taipuma oikeanpuoleisten tukien keskellä
The deflection at the midpoint between the right supports

Element 2

$$H_1 = \frac{1}{4}(2 - 3\xi + \xi^3) \quad , \quad H_2 = \frac{1}{4}(1 - \xi - \xi^2 + \xi^3)$$
$$H_3 = \frac{1}{4}(2 + 3\xi - \xi^3) \quad , \quad H_4 = \frac{1}{4}(-1 - \xi + \xi^2 + \xi^3)$$
$$v(\xi) = H_1 q_1 + \frac{l_e}{2} H_2 q_2 + H_3 q_3 + \frac{l_e}{2} H_4 q_4$$

$$q_1 = q_3 = q_4 = 0 \quad , \quad q_2 = Q_2$$

$\Rightarrow$

$$v(0) = \frac{L}{2} \cdot H_2(0) \cdot Q_2 = \frac{L}{2} \cdot \frac{1}{4} \cdot \frac{6F \cdot L^2}{60EI} = \frac{6FL^3}{480EI} \approx 0{,}075\ \text{mm}$$

$\Rightarrow$

$$v(0) \approx 0{,}075\ \text{mm} \qquad (\text{"Exact"}\ \ 0{,}075\ \text{mm}) \qquad \leftarrow$$

Taivutusmomentti liukutuen kohdalla ; The bending moment of the sliding support

Element 1 ; Node 1

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix} = [k_1] \cdot \{Q\} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ \mathbf{6L} & \mathbf{4L^2} & \mathbf{-6L} & \mathbf{2L^2} \\ -12 & -6L & 12 & -6L \\ 6L & 2\mathrm{L}^2 & -6L & 4L^2 \end{bmatrix} \cdot \begin{Bmatrix} \mathbf{Q}_1 \\ 0 \\ 0 \\ \mathbf{Q_2} \end{Bmatrix}$$

$\Rightarrow$

$$M_1 = -R_2 = -\frac{EI}{L^3} \cdot \left(6L \cdot Q_1 + 2L^2 \cdot Q_2\right) = -\frac{EI}{L^3} \cdot \frac{L}{60EI} \cdot \left[6L \cdot \left(-F \cdot 8L^2\right) + 2L^2 \cdot \left(F \cdot 6L\right)\right]$$

$\Rightarrow$

$$M_1 = -\frac{L}{60} \cdot \left[6 \cdot \left(-F \cdot 8\right) + 2 \cdot \left(F \cdot 6\right)\right] = \frac{2 \cdot 36 \cdot 15000}{60} = 18 \text{ kNm} \qquad \longleftarrow\longleftarrow$$

Choose basis functions $\mathbf{G}_i$. Determine the coefficients $\mathbf{Q}_i$ in $\tilde{\mathbf{u}} = \sum_{i=1}^{n} \mathbf{Q}_i \mathbf{G}_i$, such that

$\int_V \phi(L\tilde{u} - P)\, dV = 0$ for every ϕ of the type $\phi = \sum_{i=1}^{n} \phi_i G_i$.

$$k_e = \frac{EA_e}{l_e}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix} \quad , \quad l = \frac{x_2 - x_1}{l_e} \quad , \quad m = \frac{y_2 - y_1}{l_e}$$

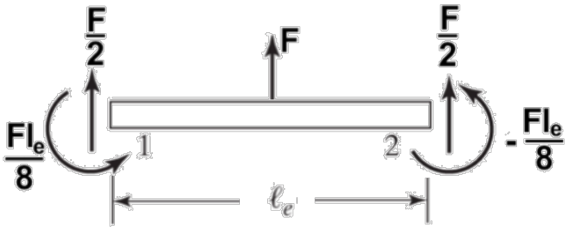
$$\sigma = \frac{E_e}{l_e}\begin{bmatrix} -l & -m & l & m \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

$$k_e = \frac{EI}{l_e^3}\begin{bmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{bmatrix}$$

$$H_1 = \frac{1}{4}(2 - 3\xi + \xi^3) \quad , \quad H_2 = \frac{1}{4}(1 - \xi - \xi^2 + \xi^3)$$

$$H_3 = \frac{1}{4}(2 + 3\xi - \xi^3) \quad , \quad H_4 = \frac{1}{4}(-1 - \xi + \xi^2 + \xi^3)$$

$$v(\xi) = H_1 q_1 + \frac{l_e}{2} H_2 q_2 + H_3 q_3 + \frac{l_e}{2} H_4 q_4$$



$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \cdot \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} - \begin{Bmatrix} F/2 \\ FL/8 \\ F/2 \\ -FL/8 \end{Bmatrix}$$