TTY / Talous ja rakentaminen

Rakennustekniikka

RAK-32300 ELEMENTTIMENETELMÄN PERUSTEET, 4 op

Syksy 2018 Jarmo Poutala

1. Tentti ratkaisut

ti 18.12.2018

Välikoe 1	Välikokeeseen 1 kuuluvat tehtävät 1, 2 ja 5. (Merkitse 1V)
Välikoe 2	Välikokeeseen 2 kuuluvat tehtävät 3, 4 ja 6. (Merkitse 2V)
Tentti	Tenttiin kuuluvat tehtävät 1-4. (Merkitse T)

1. Ratkaise Galerkinin menetelmällä differentiaaliyhtälö

$$\frac{du}{dx} + 5\cdot u = 4\cdot x^2 + 3\cdot x, \qquad x \in [0,1], \qquad u(0) = 0$$

käyttäen vain yhtä kantafunktiota $\mathbf{G_1} = \mathbf{x}\cdot(\mathbf{x}+\mathbf{0{,}2})$. Vertaa saatua tulosta tarkkaan ratkaisuun, kun **x=0,5**.

$$u(x)_{Exact} = \frac{7\cdot x}{25} + \frac{7\cdot e^{-5\cdot x}}{125} + \frac{4\cdot x^2}{5} - \frac{7}{125}$$

Basis function $\mathrm{G}_1(x) = x\cdot(x+0{,}2)$

$\Rightarrow$

$$\tilde{u}(x) = Q_1\cdot x\cdot(x+0{,}2) = Q_1\cdot x^2 + 0{,}2\cdot Q_1\cdot x$$

$$\tilde{u}'(x) = 2\cdot Q_1\cdot x + 0{,}2\cdot Q_1$$

$\Rightarrow$

$$L\tilde{u} - P = 2\cdot Q_1\cdot x + 0{,}2\cdot Q_1 + 5\cdot Q_1\cdot x^2 + 1{,}0\cdot Q_1\cdot x - 4\cdot x^2 - 3\cdot x$$

$$\phi = \phi_1\cdot(x^2 + 0{,}2\cdot x)$$

$\Rightarrow$

$$\int_0^1 \phi\cdot(L\tilde{u} - P)\cdot dx = 0 \qquad \forall \phi_i$$

$\Rightarrow$ $\downarrow$ Valitse ; Choose $\downarrow$

$$\int_0^1 (x^2 + 0{,}2\cdot x)\cdot\left(2\cdot Q_1\cdot x + 0{,}2\cdot Q_1 + 5\cdot Q_1\cdot x^2 + 1{,}0\cdot Q_1\cdot x - 4\cdot x^2 - 3\cdot x\right)\cdot dx = 0 \qquad \leftarrow (\phi_1 = 1)$$

$\Rightarrow$

$$\int_0^1 (3\cdot Q_1 x^3 + 0{,}2\cdot Q_1 x^2 + 5\cdot Q_1 x^4 - 4\cdot x^4 - 3\cdot x^3 \quad \ldots$$

$$+0{,}6\cdot Q_1 x^2 + 0{,}2^2\cdot Q_1 x + 1{,}0\cdot Q_1 x^3 - 0{,}8\cdot x^3 - 0{,}6\cdot x^2)\cdot dx = 0$$

$$\int_0^1 (3\cdot Q_1 x^3 + 0,2\cdot Q_1 x^2 + 5\cdot Q_1 x^4 - 4\cdot x^4 - 3\cdot x^3 \quad \ldots$$

$$+0,6\cdot Q_1 x^2 + 0,2^2\cdot Q_1 x + 1,0\cdot Q_1 x^3 - 0,8\cdot x^3 - 0,6\cdot x^2)\cdot dx = 0$$

$\Rightarrow$

$$\Big/_0^1 ((3\cdot Q_1 \frac{x^4}{4} + 0,2\cdot Q_1 \frac{x^3}{3} + 5\cdot Q_1 \frac{x^5}{5} - 4\frac{x^5}{5} - 3\frac{x^4}{4} \quad \ldots$$

$$+0,6\cdot Q_1 \frac{x^3}{3} + 0,2^2\cdot Q_1 \frac{x^2}{2} + 1,0\cdot Q_1 \frac{x^4}{4} - 0,8\cdot \frac{x^4}{4} - 0,6\cdot \frac{x^3}{3}) = 0$$

$\Rightarrow$

$$\left(\frac{3}{4} + \frac{0,2}{3} + \frac{5}{5} + \frac{0,6}{3} + \frac{0,04}{2} + \frac{1}{4}\right)\cdot Q_1 = \left(\frac{4}{5} + \frac{3}{4} + \frac{0,8}{4} + \frac{0,6}{3}\right)$$

$\Rightarrow$

$$Q_1 = \frac{585}{686} \approx 0,8528$$

$\Rightarrow$

Solution

$$\tilde{u}(x) = \frac{585}{686}\cdot x\cdot (x+0,2)$$

$$\tilde{u}(0,5) = \frac{585}{686}\cdot x\cdot (x+0,2) = \frac{585}{686}\cdot 0,5\cdot (0,5+0,2) = \frac{117}{392} \approx 0,2985 \qquad \leftarrow\leftarrow (1)$$

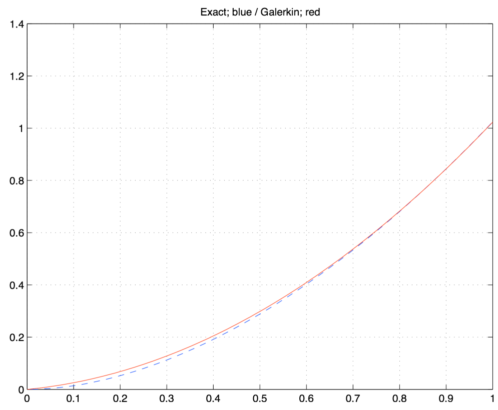
Exact solution

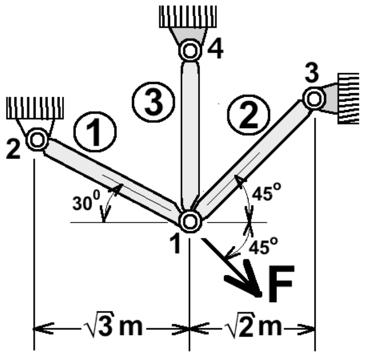
$$u(x)_{Exact} = \frac{7\cdot x}{25} + \frac{7\cdot e^{-5\cdot x}}{125} + \frac{4\cdot x^2}{5} - \frac{7}{125}$$

$$u(0,5)_{Exact} = \frac{7\cdot 0,5}{25} + \frac{7\cdot e^{-5\cdot 0,5}}{125} + \frac{4\cdot 0,5^2}{5} - \frac{7}{125} \approx 0,2886 \qquad \leftarrow\leftarrow (2)$$

$\Rightarrow \qquad (x = 0,5)$

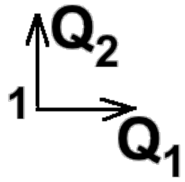
The Galerkin approximate solution (1) $\cong$ The exact analytical solution of the differential equation (2)





2. Määritä kuvassa olevan kolmisauvaisen ristikon sekä solmun **1** siirtymät että sauvojen **1** ja **3** normaalijännitykset elementtimenetelmällä. Sauva **1** on vaakasuoraan nähden kulmassa **30°**. Sauva **2** on vaakasuoraan nähden kulmassa **45°**. Sauva **3** on pystysuorassa. Sauvojen pituus **L=2 m**. Materiaalin kimmomoduuli **E=200 GPa** ja sauvojen poikkipinta-ala **A=600 mm²** . Solmuun **1** vaikuttaa vaakasuoraan nähden kulmassa **45°** oleva voima $\mathbf{F} = \mathbf{75000 \cdot \sqrt{2}\ N}$.

$$\sigma = \frac{E_e}{\ell_e}\begin{bmatrix} -\ell & -m & \ell & m \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$



$$E = 200\ \text{GPa},\ A = 600 mm^2, \quad F = 75000 \cdot \sqrt{2}\ N, \quad L = 2\ m$$

Sauva 1 , Element 1

$$A_1 = 600 mm^2 = 600 \cdot 10^{-6} m^2, \quad 2 \to 1 \qquad l_1 = \frac{\sqrt{3}}{2}, \quad m_1 = -\frac{1}{2}, \quad L_1 = 2\ m = L,$$

Sauva 2 , Element 2

$$A_2 = 600 mm^2 = 600 \cdot 10^{-6} m^2, \quad 3 \to 1 \qquad l_2 = -\frac{\sqrt{2}}{2}, \quad m_2 = -\frac{\sqrt{2}}{2}, \quad L_2 = 2\ m = L,$$

Sauva 3 , Element 3

$$A_3 = 600 mm^2 = 600 \cdot 10^{-6} m^2, \quad 4 \to 1 \qquad l_3 = 0, \quad m_3 = -1, \quad L_3 = 2\ m = L,$$

Kaksi vapausastetta 2 DOF , (1 horizontal Q_1 , 2 vertical Q_2) , Node 1

$$k_e = \frac{EA}{l_e}\begin{bmatrix} l^2 & lm \\ lm & m^2 \end{bmatrix}, \quad k_1 = \frac{EA}{L}\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 3/4 & -\sqrt{3}/4 \\ -\sqrt{3}/4 & 1/4 \end{bmatrix} \end{array}\begin{array}{c} 1 \\ 2 \end{array}, \quad k_2 = \frac{EA}{L}\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 2/4 & 2/4 \\ 2/4 & 2/4 \end{bmatrix} \end{array}\begin{array}{c} 1 \\ 2 \end{array}, \quad k_3 = \frac{EA}{L}\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{array}\begin{array}{c} 1 \\ 2 \end{array}$$

$\Rightarrow$

$$[K] = \sum_{i=1}^{3} k_i = \frac{EA}{L}\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 5/4 & 2/4 - \sqrt{3}/4 \\ 2/4 - \sqrt{3}/4 & 7/4 \end{bmatrix} \end{array}\begin{array}{c} 1 \\ 2 \end{array} \qquad \{F\} = \begin{Bmatrix} 75000 \\ -75000 \end{Bmatrix}\begin{array}{c} 1 \\ 2 \end{array}$$

$\Rightarrow$

$$[K]\{Q\} = \{F\} \qquad \det = 5 \cdot 7/16 - \left(2/4 - \sqrt{3}/4\right)^2$$

$\Rightarrow$

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = [K]^{-1}\{F\} = \frac{L}{EA} \cdot \frac{1}{\det}\begin{bmatrix} 7/4 & \sqrt{3}/4 - 2/4 \\ \sqrt{3}/4 - 2/4 & 5/4 \end{bmatrix}\begin{Bmatrix} 75000 \\ -75000 \end{Bmatrix} =$$

$\rightarrow$

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \frac{2000}{200 \cdot 10^3 \cdot 600 \cdot \det} \cdot \begin{Bmatrix} (7/4) \cdot (75000) - (\sqrt{3}/4 - 2/4) \cdot 75000 \\ (\sqrt{3}/4 - 2/4) \cdot 75000 - (5/4) \cdot 75000 \end{Bmatrix} \approx \begin{Bmatrix} 1{,}0404 \\ -0{,}7541 \end{Bmatrix} \text{mm}$$

Stress in the element 1

Node 2 fixed $\Rightarrow$ $q_1 = q_2 = 0$, $q_3 = Q_1$, $q_4 = Q_2$ *Node* 2 $\rightarrow$ Node 1

$$\sigma_1 = \frac{E}{L_1}\begin{bmatrix}-l_1 & -m_1 & l_1 & m_1\end{bmatrix}\begin{Bmatrix}q_1\\q_2\\q_3\\q_4\end{Bmatrix} = \frac{200\cdot 10^3}{2000}\left[\frac{\sqrt{3}}{2}\cdot(1{,}04) - \frac{1}{2}\cdot(-0{,}7541)\right] \approx 127{,}8\ \text{MPa}$$

$\Rightarrow$

$\sigma_1 = 128$ MPa

Stress in the element 2

Node 3 fixed $\Rightarrow$ $q_1 = q_2 = 0$, $q_3 = Q_1$, $q_4 = Q_2$ *Node* 3 $\rightarrow$ Node 1

$$\sigma_2 = \frac{E}{L_2}\begin{bmatrix}-l_2 & -m_2 & l_2 & m_2\end{bmatrix}\begin{Bmatrix}q_1\\q_2\\q_3\\q_4\end{Bmatrix} = \frac{200\cdot 10^3}{2000}\left[(-\frac{\sqrt{2}}{2})\cdot(1{,}04) + (-\frac{\sqrt{2}}{2})\cdot(-0{,}7541)\right] \approx -20{,}2\ \text{MPa}$$

$\Rightarrow$

$\sigma_2 = -20$ MPa

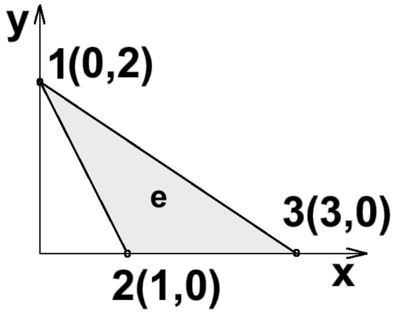
Stress in the element 3

Node 4 fixed $\Rightarrow$ $q_1 = q_2 = 0$, $q_3 = Q_1$, $q_4 = Q_2$ *Node* 4 $\rightarrow$ Node 1

$$\sigma_3 = \frac{E}{L_3}\begin{bmatrix}-l_3 & -m_3 & l_3 & m_3\end{bmatrix}\begin{Bmatrix}q_1\\q_2\\q_3\\q_4\end{Bmatrix} = \frac{20\cdot 10^3}{2000}\left[0\cdot(1{,}04) - 1\cdot(-0{,}7541)\right] \approx 75{,}4\ \text{MPa}$$

$\Rightarrow$

$\sigma_3 = 75$ MPa



3. Ratkaise integraali $I_{xx} = \iint\limits_A x^2 \, dA$ alueessa **e**

a) käyttäen kolmisolmuista **CST** elementtiä ja kuvan alla olevaa kaavaa ja

b) käyttäen Gaussin numeerista integrointia ja nelisolmuista elementtiä, jolloin solmut 3 ja 4 yhtenevät.

$$N_1 = \xi \qquad N_2 = \eta \qquad N_3 = 1 - \xi - \eta$$

$$I = \iint\limits_A f(x,y)dA = \int_0^1 \int_0^{1-\xi} f(\xi,\eta)\det(J)d\xi d\eta$$

$$I = \int_0^1 \int_0^{1-\xi} f(\xi,\eta)\det(J)\cdot d\xi d\eta = \int_0^1 \int_0^{1-\xi} N_1^a \cdot N_2^b \cdot N_3^c \cdot 2A \cdot d\xi d\eta$$

$$I = \int_0^1 \int_0^{1-\xi} N_1^a \cdot N_2^b \cdot N_3^c \cdot 2A \cdot d\xi d\eta = \frac{a!\ b!\ c!}{(a+b+c+2)!} \cdot 2A$$

Solmukoordinaatit

$$\begin{cases} x_1 = 0 \quad , \quad x_2 = 1 \quad , \quad x_3 = 3 \quad , \quad x_4 = x_3 = 3 \\ y_1 = 2 \quad , \quad y_2 = 0 \quad , \quad y_3 = 0 \quad , \quad y_4 = y_3 = 0 \end{cases}$$

Nodal coordinates

CST - elementti

CST element

$$\det(J) = 2A = 2 \cdot \left(\frac{2 \cdot 2}{2} \right) = 4$$

$$x = N_1 \cdot x_1 + N_2 \cdot x_2 + N_3 \cdot x_3 = N_1 \cdot 0 + N_2 \cdot 1 + N_3 \cdot 3 = N_2 + 3 \cdot N_3$$

$$y = N_1 \cdot y_1 + N_2 \cdot y_2 + N_3 \cdot y_3 = N_1 \cdot 2 + N_2 \cdot 0 + N_3 \cdot 0 = 2 \cdot N_1$$

$\Rightarrow$

$$x^2 = \left(N_2 + 3 \cdot N_3\right)^2 = N_2^2 + 6 \cdot N_2 \cdot N_3 + 9 \cdot N_3^2$$

$\Rightarrow$

$$I_1 = \int_0^1 \int_0^{1-\xi} N_1^a \cdot N_2^b \cdot N_3^c \cdot 2A \cdot d\xi d\eta = \int_0^1 \int_0^{1-\xi} N_1^0 \cdot N_2^2 \cdot N_3^0 \cdot 2A \cdot d\xi d\eta = \frac{0!\ 2!\ 0!}{(0+2+0+2)!} \cdot 4 = \frac{1}{3} \qquad \leftarrow$$

$$I_2 = \int_0^1 \int_0^{1-\xi} N_1^a \cdot N_2^b \cdot N_3^c \cdot 2A \cdot d\xi d\eta = \int_0^1 \int_0^{1-\xi} N_1^0 \cdot 6 \cdot N_2^1 \cdot N_3^1 \cdot 2A \cdot d\xi d\eta = 6 \cdot \frac{0!\ 1!\ 1!}{(0+1+1+2)!} \cdot 4 = 1 \qquad \leftarrow$$

$$I_3 = \int_0^1 \int_0^{1-\xi} N_1^a \cdot N_2^b \cdot N_3^c \cdot 2A \cdot d\xi d\eta = \int_0^1 \int_0^{1-\xi} N_1^0 \cdot N_2^0 \cdot 9 \cdot N_3^2 \cdot 2A \cdot d\xi d\eta = 9 \cdot \frac{0!\ 0!\ 2!}{(0+0+2+2)!} \cdot 4 = \frac{9}{3} \qquad \leftarrow$$

$\Rightarrow$

$$I = I_1 + I_2 + I_3 = \frac{1}{3} + 1 + \frac{9}{3} = \frac{13}{3} \approx 4{,}3333 \qquad \leftarrow\leftarrow\leftarrow$$

4 - solmuinen 4 - sivuinen elementti 4 - node quadrilateral element

$$x = N_1 \cdot x_1 + N_2 \cdot x_2 + N_3 \cdot x_3 + N_4 \cdot x_4 = N_1 \cdot 0 + N_2 \cdot 1 + N_3 \cdot 3 + N_4 \cdot 3 \ldots$$

$$= \frac{(1+\xi)(1-\eta)}{4} + \frac{(1+\xi)(1+\eta)\cdot 3}{4} + \frac{(1-\xi)(1+\eta)\cdot 3}{4} = \frac{1}{4}\cdot(1+\xi-\eta-\xi\cdot\eta+6+6\cdot\eta)$$

$$x = \frac{1}{4}\cdot(7+\xi+5\cdot\eta-\xi\cdot\eta)$$

$\Rightarrow$

$$y = N_1 \cdot y_1 + N_2 \cdot y_2 + N_3 \cdot y_3 + N_4 \cdot y_4 = N_1 \cdot 2 + N_2 \cdot 0 + N_3 \cdot 0 + N_4 \cdot 0$$

$$\Rightarrow \qquad y = \frac{(1-\xi)(1-\eta)\cdot 2}{4} = 2\cdot\frac{(1-\xi)(1-\eta)}{4}$$

$\Rightarrow$

Jacobin matriisi ja sen determinantti The Jacobian and its determinant

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \frac{1}{4}\cdot\begin{bmatrix} 1-\eta & -2\cdot(1-\eta) \\ (5-\xi) & -2\cdot(1-\xi) \end{bmatrix}$$

$$\det(J) = \left(\frac{1}{4}\right)^2 \cdot (-2)\cdot(1-\eta)\cdot(1-\xi) + 2\cdot(5-\xi)\cdot(1-\eta)$$

$\Rightarrow$

$$I = \iint x^2 dA = \int_{-1}^{1}\int_{-1}^{1}\left(\frac{1}{4}\cdot(7+\xi+5\cdot\eta-\xi\cdot\eta)\right)^2 \det(J)\cdot d\xi\cdot d\eta$$

$\Rightarrow$

$$I = \int_{-1}^{1}\int_{-1}^{1}\left(\frac{1}{4}\cdot(7+\xi+5\cdot\eta-\xi\cdot\eta)\right)^2\cdot\left(\frac{1}{4}\right)^2\cdot\left((-2)\cdot(1-\eta)\cdot(1-\xi)+2\cdot(5-\xi)\cdot(1-\eta)\right)\cdot d\xi\cdot d\eta$$

$\Rightarrow$

$$I = \int_{-1}^{1}\int_{-1}^{1}\left(\frac{1}{4}\cdot(7+\xi+5\cdot\eta-\xi\cdot\eta)\right)^2\cdot\left(\frac{1}{4}\right)^2\cdot(8-8\cdot\eta)\cdot d\xi\cdot d\eta$$

$\Rightarrow$

$$I=\int_{-1}^{1}\int_{-1}^{1}\left(\frac{1}{4}\cdot\left(7+\xi+5\cdot\eta-\xi\cdot\eta\right)\right)^{2}\cdot\left(\frac{1}{4}\right)^{2}\cdot\left(8-8\cdot\eta\right)\cdot d\xi\cdot d\eta$$

$2N-1=2\cdot 2-1=3 \qquad \Rightarrow 2$ - point integration is exact

$\Rightarrow$

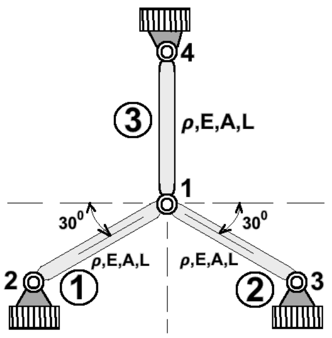
$$f(\xi,\eta)=\frac{1}{32}\cdot\left(7+\xi+5\cdot\eta-\xi\cdot\eta\right)^{2}\cdot\left(1-\eta\right)$$

Gaussian quadrature

$$\mathrm{I}=\sum_{i=1}^{2}\sum_{j=1}^{2}w_i\cdot w_j\cdot f(\xi_i,\eta_j)=f(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}})+f(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}})+f(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}})+f(-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}})$$

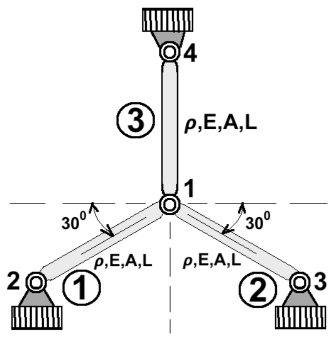
$\Rightarrow$

$$\mathrm{I}=0{,}5056+1{,}2281+1{,}2441+1{,}3555=\frac{13}{3}\approx 4{,}3333 \qquad \leftarrow$$

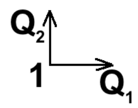


4. Ratkaise elementtimenetelmällä kuvan ristikkorakenteen alimmat ominaistaajuudet käyttäen keskitettyä massamatriisia. Sauvojen poikkipinta-ala on **A** . Sauvojen pituus **L=3 m**. Materiaalina olevan teräksen tiheys on ρ.

$$E = 200 \text{ GPa},\ A = 5{,}38\cdot 10^3 \text{ mm}^2,\ \rho = 7844\ \frac{\text{kg}}{\text{m}^3}$$



$$E = 200 \text{ GPa},\ A = 5{,}38\cdot 10^3\ mm^2, \quad \rho = 7844\ \frac{kg}{m^3}, \quad L = 3 \text{ m}$$



Sauva 1 , Element 1

$$A_1 = A = 5{,}38\cdot 10^{-3}\ m^2, \quad 2 \rightarrow 1 \qquad l_1 = \frac{\sqrt{3}}{2}, \quad m_1 = \frac{1}{2}, \quad L_1 = L = 3\ m$$

Sauva 2 , Element 2

$$A_2 = A = 5{,}38\cdot 10^{-3}\ m^2, \quad 3 \rightarrow 1 \qquad l_2 = -\frac{\sqrt{3}}{2}, \quad m_2 = \frac{1}{2}, \quad L_2 = \mathrm{L} = 3\ m$$

Sauva 3 , Element 3

$$A_3 = A = 5{,}38\cdot 10^{-3}\ m^2, \quad 4 \rightarrow 1 \qquad l_3 = 0, \quad m_3 = -1, \quad L_3 = \mathrm{L} = 3\ m$$

Kaksi vapausastetta 2 DOF , (1 horizontal Q_1 , 2 vertical Q_2) , Node 1

Jäykkyysmatriisi :: Stiffness matrix $[\mathrm{K}]$

$$k_e = \frac{EA}{l_e}\begin{bmatrix} l^2 & lm \\ lm & m^2 \end{bmatrix}, \quad k_1 = \frac{EA}{L}\begin{array}{cc} \begin{array}{cc} 1 & 2 \end{array} & \\ \begin{bmatrix} 3/4 & \sqrt{3}/4 \\ \sqrt{3}/4 & 1/4 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array}, \quad k_2 = \frac{EA}{L}\begin{array}{cc} \begin{array}{cc} 1 & 2 \end{array} & \\ \begin{bmatrix} 3/4 & -\sqrt{3}/4 \\ -\sqrt{3}/4 & 1/4 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array}, \quad k_3 = \frac{EA}{L}\begin{array}{cc} \begin{array}{cc} 1 & 2 \end{array} & \\ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array}$$

$\Rightarrow$

$$[K] = \sum_{i=1}^{2} k_i = \frac{EA}{L}\cdot\begin{bmatrix} 3/2 & 0 \\ 0 & 3/2 \end{bmatrix} = \frac{EA}{L}\cdot\begin{bmatrix} 3/2 & 0 \\ 0 & 3/2 \end{bmatrix}$$

$\Rightarrow$

Keskitetty massamatriisi Lumped mass matrix

$$m_1 = \frac{\rho AL}{2}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \frac{\rho A_1 L_1}{2}\begin{array}{c}\begin{array}{cc}1 & 2\end{array}\\ \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\end{array}\begin{array}{c}\\1\\2\end{array},\quad m_2 = \frac{\rho A_2 L_2}{2}\begin{array}{c}\begin{array}{cc}1 & 2\end{array}\\ \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\end{array}\begin{array}{c}\\1\\2\end{array},\quad m_3 = \frac{\rho A_3 L_2}{2}\begin{array}{c}\begin{array}{cc}1 & 2\end{array}\\ \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\end{array}\begin{array}{c}\\1\\2\end{array}$$

$\Rightarrow$

$$[\mathrm{M}] = \sum_{i=1}^{3} m_i = \frac{\rho AL}{2}\begin{bmatrix}3 & 0\\0 & 3\end{bmatrix}$$

Karakteristinen yhtälö ; Characteristic polynomial

$$\det([K] - \lambda\cdot[\mathrm{M}]) = 0 \quad , \quad \lambda = \omega^2 \quad \Rightarrow \quad \omega = \sqrt{\lambda} \quad \Rightarrow \quad f = \frac{\omega}{2\cdot\pi}$$

$$\det\left(\frac{EA}{L}\begin{bmatrix}3/2 & 0\\0 & 3/2\end{bmatrix} - \lambda\cdot\frac{\rho AL}{2}\begin{bmatrix}3 & 0\\0 & 3\end{bmatrix}\right) = 0 \qquad \Rightarrow \qquad \bar{\lambda} = \frac{\rho AL^2}{EA}\lambda$$

$\Rightarrow$

$$\det\begin{pmatrix}1 - 1\cdot\bar{\lambda} & 0\\0 & 1 - 1\cdot\bar{\lambda}\end{pmatrix} = 0$$

$\Rightarrow$

$$\left(1 - \bar{\lambda}\right)\cdot\left(1 - \bar{\lambda}\right) = 0$$

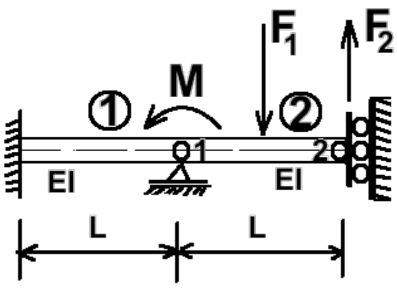
$\Rightarrow$

$$\Rightarrow \qquad \begin{cases}\bar{\lambda}_1 = 1\\ \bar{\lambda}_2 = 1\end{cases}$$

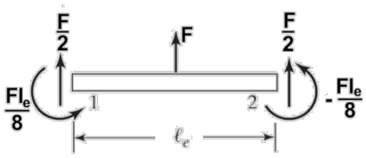
Alimmat ominaistaajuudet , Lowest natural frequencies

$$f_1 = \frac{\omega_1}{2\cdot\pi} = \frac{\sqrt{\lambda_1}}{2\cdot\pi} = \frac{1}{2\cdot\pi}\sqrt{\frac{EA}{\rho AL^2}\cdot\bar{\lambda}_1} \approx 267{,}88 \text{ Hz}, \qquad f_2 = \frac{\omega_2}{2\cdot\pi} = \frac{\sqrt{\lambda_2}}{2\cdot\pi} = \frac{1}{2\cdot\pi}\sqrt{\frac{EA}{\rho AL^2}\cdot\bar{\lambda}_2} \approx 267{,}88 \text{ Hz}$$

(Ansys Classic : $\mathrm{f}_1 = 267{,}88$ Hz , $\mathrm{f}_2 = 267{,}88$ Hz , 3 elements, 2 DOF)

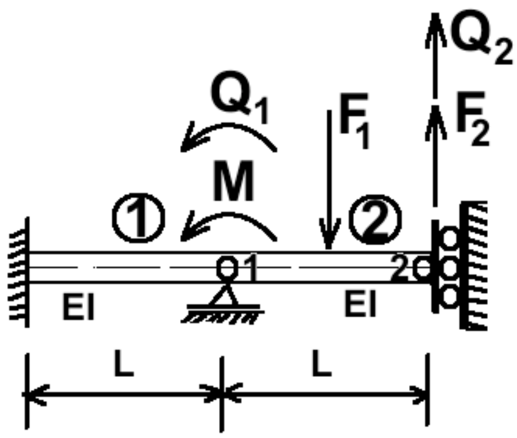


5. Määritä kuvassa olevan kahdesta palkkielementistä muodostuvan tasokehän solmun **1** kiertymä ja solmun **2** siirtymä sekä elementin **1** keskipisteen taipuma elementtimenetelmällä. Elementit oletetaan venymättömiksi. Solmussa **1** vaikuttaa taivutusmomentti **M=100 kNm** ja solmussa **2** on pistevoima $\mathbf{F_2}$**. =50 kN**. Elementtien pituus **L=2 m**. Materiaalin kimmomoduuli **E=200 GPa** ja palkkien poikkipinnan neliömomentti $\mathbf{I=10^{-4}\ m^4}$. Elementin **2** keskellä vaikuttaa pistevoima $\mathbf{F_1}$**=100 kN** alaspäin.



$$v(\xi) = H_1 q_1 + \frac{l_e}{2} H_2 q_2 + H_3 q_3 + \frac{l_e}{2} H_4 q_4$$

$$F_1 = \frac{F}{2}, \quad F_2 = \frac{Fl_e}{8}, \quad F_3 = \frac{F}{2}, \quad F_4 = -\frac{Fl_e}{8}$$



Jäykkyysmatriisi Stiffness matrix

Palkki Beam

$$k_e = \frac{EI_Z}{l_e^3}\begin{bmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{bmatrix}$$

$$[k_1] = \frac{EI}{L^3}\begin{array}{c} \begin{array}{cccc} & & & 1 \end{array} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & \mathbf{4L^2} \end{bmatrix} \begin{array}{c} \\ \\ \\ 1 \end{array} \end{array}, \quad [k_2] = \frac{EI}{L^3}\begin{array}{c} \begin{array}{cccc} & 1 & 2 & \end{array} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & \mathbf{4L^2} & \mathbf{-6L} & 2L^2 \\ -12 & \mathbf{-6L} & \mathbf{12} & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{array}{c} \\ 1 \\ 2 \\ \\ \end{array} \end{array}$$

$\Rightarrow$

After elimination

$$[\mathrm{K}] = \frac{EI}{L^3} \begin{array}{cc|c} 1 & 2 & \\ \left[\begin{array}{cc} 8L^2 & -6L \\ -6L & 12 \end{array}\right] & & \begin{array}{c} 1 \\ 2 \end{array} \end{array} \quad , \quad \det = 8L^2 \cdot 12 - (-6L) \cdot (-6L) = 60L^2$$

Loading vector Equivalent nodal moment and force

Node 1 and 2 Element 2 (Nodes 1 and 2)

$$\{\mathrm{F}\} = \begin{Bmatrix} F_{Node\_1} \\ F_{Node\_2} \end{Bmatrix} = \begin{Bmatrix} M - \dfrac{F_1 L}{8} \\ -\dfrac{F_1}{2} + F_2 \end{Bmatrix} = \begin{Bmatrix} 100000 - \dfrac{100000 \cdot 2}{8} \\ -\dfrac{100000}{2} + 50000 \end{Bmatrix} = \begin{Bmatrix} 75000 \\ 0 \end{Bmatrix} \mathrm{Nm}$$

$\Rightarrow$

Kiertymä Q_1 ja siirtymä Q_2 Slope Q_1 and displacement Q_2

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = [K]^{-1} \cdot \{F\} = \frac{L^3}{EI} \cdot \frac{1}{\det} \cdot \begin{bmatrix} 12 & 6L \\ 6L & 8L^2 \end{bmatrix} \cdot \begin{Bmatrix} 75000 \\ 0 \end{Bmatrix} = \frac{75000 \cdot L^3}{60 \cdot E \cdot I} \cdot \begin{Bmatrix} 12 \\ 6L \end{Bmatrix}$$

$\Rightarrow$

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \frac{75000 \cdot L^3}{60 \cdot L^2 \cdot E \cdot I} \cdot \begin{Bmatrix} 12 \\ 6L \end{Bmatrix} = \frac{75000 \cdot 2}{60 \cdot 200 \cdot 10^9 \cdot 10^{-4}} \cdot \begin{Bmatrix} 12 \\ 6 \cdot 2 \end{Bmatrix} = \begin{Bmatrix} 0{,}0015 \\ 0{,}0015 \end{Bmatrix} = \begin{Bmatrix} 0{,}0859° \\ 1{,}5 \text{ mm} \end{Bmatrix} \qquad \leftarrow$$

Taipuma elementin 1 keskellä

The deflection at the midpoint of the element 1

Element 1

$$H_1 = \frac{1}{4}(2 - 3\xi + \xi^3) \quad , \quad H_2 = \frac{1}{4}(1 - \xi - \xi^2 + \xi^3)$$

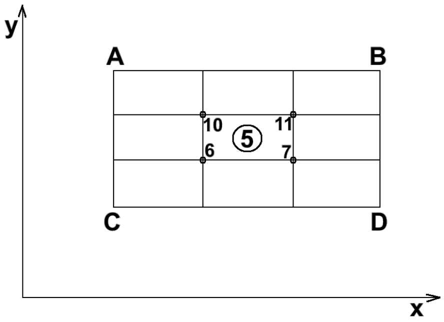
$$H_3 = \frac{1}{4}(2 + 3\xi - \xi^3) \quad , \quad H_4 = \frac{1}{4}(-1 - \xi + \xi^2 + \xi^3)$$

$$v(\xi) = H_1 q_1 + \frac{l_e}{2} H_2 q_2 + H_3 q_3 + \frac{l_e}{2} H_4 q_4$$

$$q_1 = q_2 = q_3 = 0 \quad , \quad q_4 = Q_1$$

$\Rightarrow$

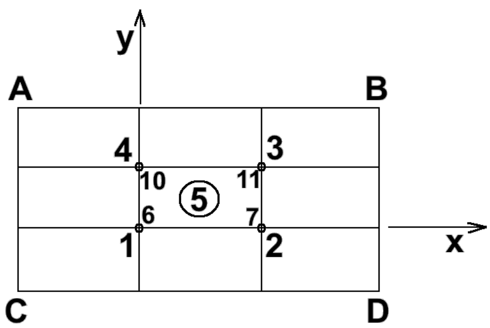
$$v(0) = \frac{L}{2} \cdot H_4(0) \cdot Q_1 = -\frac{2}{2} \cdot \frac{1}{4} \cdot Q_1 = -0{,}375 \text{ mm} \qquad \leftarrow$$



6. Määritä kuvassa olevan levyrakenteen **ABCD** elementin **5** tasojännitystilan mukaiset jännitykset tämän elementin keskipisteessä. FEM-ohjelmalla saadut nelisolmuisen elementin **5** siirtymät ovat alla olevassa taulukossa. Taulukon siirtymät ovat millimetreinä ja koordinaatit metreinä. Elementin **5** leveys on **1 m** ja korkeus on **0,5 m**. Rakenne on jäykästi kiinni solmukohdissa **6** ja **10**. Alla on kuvan tehtävään liittyvät matriisit. Materiaalin kimmomoduuli **E=200GPa** ja Poissonin vakio **ν=0,3**.

$$\begin{bmatrix} \underline{\text{Node}} & \underline{x} & \underline{y} & \underline{u} & \underline{v} \\ 6 & 2 & 1{,}5 & 0 & 0 \\ 7 & 3 & 1{,}5 & 0{,}3 & 0{,}1 \\ 10 & 2 & 2 & 0 & 0 \\ 11 & 3 & 2 & -0{,}4 & 0{,}4 \end{bmatrix} \quad , \quad \begin{Bmatrix} \varepsilon_x \\ \gamma_x \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} = [J]^{-1} \cdot \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix} \quad , \quad \begin{Bmatrix} \gamma_y \\ \varepsilon_y \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial y} \end{Bmatrix} = [J]^{-1} \cdot \begin{Bmatrix} \dfrac{\partial v}{\partial \xi} \\ \dfrac{\partial v}{\partial \eta} \end{Bmatrix} \quad , \quad \{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_x + \gamma_y \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$[D] = \frac{E}{1-\nu^2} \cdot \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} \quad , \quad \{\sigma\} = [D] \cdot \{\varepsilon\}, \quad E = 200 \text{ GPA}, \ \nu = 0{,}3$$



$$\begin{array}{c} \begin{bmatrix} \underline{\text{Node}} & \underline{x} & \underline{y} & \underline{u} & \underline{v} \\ 6 & 0 & 0 & 0 & 0 \\ 7 & 1000 & 0 & 0{,}3 & 0{,}1 \\ 10 & 0 & 500 & 0 & 0 \\ 11 & 1000 & 500 & -0{,}4 & 0{,}4 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta) \quad , \quad N_2 = \frac{1}{4}(1+\xi)(1-\eta) \quad , \quad N_3 = \frac{1}{4}(1+\xi)(1+\eta) \quad , \quad N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

$$[-1 \le \xi \le 1] \quad , \quad [-1 \le \eta \le 1]$$

$\Rightarrow$

$$x = N_2 x_2 + N_3 x_3 = \frac{1}{4}\left[(1+\xi)(1-\eta) \cdot 1000 + (1+\xi)(1+\eta) \cdot 1000\right] = 500 \cdot \xi + 500$$

$$y = N_3 y_3 + N_4 y_4 = \frac{1}{4}\left[(1+\xi)(1+\eta) \cdot 500 + (1-\xi)(1+\eta) \cdot 500\right] = 250 \cdot \eta + 250$$

$$J=\begin{bmatrix}\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi}\\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta}\end{bmatrix}=\begin{bmatrix}500 & 0\\ 0 & 250\end{bmatrix}\leftarrow(\xi=\eta=0)\quad\Rightarrow\quad J=\begin{bmatrix}500 & 0\\ 0 & 250\end{bmatrix}$$

$$\det[J]=125000$$

$$[J]^{-1}=\frac{1}{125000}\begin{bmatrix}250 & 0\\ 0 & 500\end{bmatrix}=\frac{1}{500}\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}$$

$$u=N_2\cdot q_3+N_3\cdot q_5=\frac{1}{4}(1+\xi)(1-\eta)\cdot 0{,}3-\frac{1}{4}(1+\xi)(1-\eta)\cdot 0{,}4=-\frac{1}{40}((7\cdot\eta+1)\cdot(\xi+1))$$

$$\Rightarrow\frac{\partial u}{\partial \xi}=-\frac{1}{40}\cdot(7\cdot\eta+1)\qquad,\frac{\partial u}{\partial \eta}=-\frac{1}{40}\cdot(7\cdot\xi+1)$$

$$v=N_2\cdot q_4+N_3\cdot q_6=\frac{1}{4}(1+\xi)(1-\eta)\cdot 0{,}1+\frac{1}{4}(1+\xi)(1-\eta)\cdot 0{,}4=\frac{1}{40}((3\cdot\eta+5)\cdot(\xi+1))$$

$$\Rightarrow\frac{\partial v}{\partial \xi}=\frac{1}{40}\cdot(3\cdot\eta+5)\qquad,\frac{\partial v}{\partial \eta}=\frac{1}{40}\cdot(3\cdot\xi+3)$$

$$\begin{Bmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial u}{\partial y}\end{Bmatrix}=[J]^{-1}\begin{Bmatrix}\frac{\partial u}{\partial \xi}\\ \frac{\partial u}{\partial \eta}\end{Bmatrix}=\frac{1}{500}\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\cdot\begin{Bmatrix}-\frac{1}{40}\cdot(7\cdot\eta+1)\\ -\frac{1}{40}\cdot(7\cdot\xi+1)\end{Bmatrix}\leftarrow(\xi=\eta=0)\quad\Rightarrow\quad\begin{Bmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial u}{\partial y}\end{Bmatrix}=\begin{Bmatrix}-\frac{1}{20000}\\ -\frac{7}{10000}\end{Bmatrix}$$

$$\begin{Bmatrix}\frac{\partial v}{\partial x}\\ \frac{\partial v}{\partial y}\end{Bmatrix}=[J]^{-1}\begin{Bmatrix}\frac{\partial v}{\partial \xi}\\ \frac{\partial v}{\partial \eta}\end{Bmatrix}=\frac{1}{500}\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\cdot\begin{Bmatrix}\frac{1}{40}\cdot(3\cdot\eta+5)\\ \frac{1}{40}\cdot(3\cdot\xi+3)\end{Bmatrix}\leftarrow(\xi=\eta=0)\quad\Rightarrow\quad\begin{Bmatrix}\frac{\partial v}{\partial x}\\ \frac{\partial v}{\partial y}\end{Bmatrix}=\begin{Bmatrix}\frac{1}{4000}\\ \frac{3}{10000}\end{Bmatrix}$$

$\Rightarrow$

$$\begin{Bmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial u}{\partial y}\end{Bmatrix}=\begin{Bmatrix}-\frac{1}{20000}\\ -\frac{7}{10000}\end{Bmatrix}\quad,\quad\begin{Bmatrix}\frac{\partial v}{\partial x}\\ \frac{\partial v}{\partial y}\end{Bmatrix}=\begin{Bmatrix}\frac{1}{4000}\\ \frac{3}{10000}\end{Bmatrix}$$

$\Rightarrow$

$$\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}=\begin{Bmatrix}\dfrac{\partial u}{\partial x}\\ \dfrac{\partial v}{\partial y}\\ \dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x}\end{Bmatrix}=\begin{Bmatrix}-\dfrac{1}{20000}\\ \dfrac{3}{10000}\\ -\dfrac{7}{10000}+\dfrac{1}{4000}\end{Bmatrix}=\begin{Bmatrix}-\dfrac{1}{20000}\\ \dfrac{3}{10000}\\ -\dfrac{9}{20000}\end{Bmatrix}\ \frac{\mu\text{m}}{\text{m}}$$

$\Rightarrow$

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix}=\frac{E}{1-\nu^2}\begin{bmatrix}1&\nu&0\\ \nu&1&0\\ 0&0&\dfrac{1-\nu}{2}\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}=\frac{200\cdot 10^9}{1-0{,}3^2}\begin{bmatrix}1&0{,}3&0\\ 0{,}3&1&0\\ 0&0&\dfrac{1-0{,}3}{2}\end{bmatrix}\cdot\begin{Bmatrix}-\dfrac{1}{20000}\\ \dfrac{3}{10000}\\ -\dfrac{9}{20000}\end{Bmatrix}$$

$\Rightarrow$

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix}\approx\begin{Bmatrix}8{,}8\\ 62{,}6\\ -34{,}6\end{Bmatrix}\text{MPa}$$

FEM formulas **Autumn 2018**

$$N_1(\xi) = \frac{1}{2}(1-\xi) \qquad N_2(\xi) = \frac{1}{2}(1+\xi) \qquad H_1 = \frac{1}{4}(1-\xi)^2 \cdot (2+\xi), \quad H_2 = \frac{1}{4}(1-\xi)^2 \cdot (\xi+1)$$

$$N_1 = -\frac{1}{2}\xi \cdot (1-\xi),\ N_2 = \frac{1}{2}\xi \cdot (1+\xi),\ N_3 = (1+\xi)\cdot(1-\xi) \qquad H_3 = \frac{1}{4}(1+\xi)^2 \cdot (2-\xi), \quad H_4 = \frac{1}{4}(1+\xi)^2 \cdot (\xi-1)$$

$$N_1 = N_1(\xi,\eta) = \frac{1}{4}(1-\xi)(1-\eta) \qquad N_2 = N_2(\xi,\eta) = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3 = N_3(\xi,\eta) = \frac{1}{4}(1+\xi)(1+\eta) \qquad N_4 = N_4(\xi,\eta) = \frac{1}{4}(1-\xi)(1+\eta)$$

$$F = P + \sum_e (f_e + T_e) \qquad T^e = \iint_A N^T T dA \qquad f_e = \iiint_V N^T f dV \qquad K = \sum_e k_e \qquad M = \sum_e m_e$$

$$\xi,\eta \in [-1,1] \quad , \qquad \int_{-1}^{1} f(\xi) d\xi \approx \sum_{i=1}^{n} w_i f(\xi_i) \qquad x = \sum_{i=1}^{n} N_i x_i, \quad y = \sum_{i=1}^{n} N_i y_i$$

$$\begin{bmatrix} n & \xi_i & w_i \\ 1 & 0 & 2 \\ 2 & \pm\frac{1}{\sqrt{3}} & 1 \\ 3 & \begin{matrix} 0 \\ \pm\sqrt{\frac{3}{5}} \end{matrix} & \begin{matrix} \frac{8}{9} \\ \frac{5}{9} \end{matrix} \end{bmatrix} \qquad J = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \qquad \det([K] - \lambda[M]) = 0$$

$$k_e = \frac{EA_e}{\ell_e}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix} \qquad l = \frac{x_2 - x_1}{\ell_e}, \quad m = \frac{y_2 - y_1}{\ell_e}, \qquad k_e = \frac{EI_Z}{l_e^3}\begin{bmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{bmatrix}$$

$$m_{ec} = \frac{\rho A_e \ell_e}{6}\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \qquad m_{ec} = \frac{\rho A_e l_e}{420}\begin{bmatrix} 156 & 22l_e & 54 & -13l_e \\ 22l_e & 4l_e^2 & 13l_e & -3l_e^2 \\ 54 & 13l_e & 156 & -22l_e \\ -13l_e & -3l_e^2 & -22l_e & 4l_e^2 \end{bmatrix}$$

$$m_{elu} = \frac{\rho A_e \ell_e}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad m_{elu} = \frac{\rho A_e l_e}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Choose basis functions $\mathbf{G_i}$. Determine the coefficients $\mathbf{Q_i}$ in $\tilde{\mathbf{u}} = \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathbf{Q_i} \mathbf{G}_i$, such that

$\int_V \phi\left(L\tilde{u} - P\right) dV = 0$ for every ϕ of the type $\phi = \sum_{\mathrm{i}=1}^{\mathrm{n}} \phi_{\mathrm{i}} G_i$.