

MAT-31101 Numeerinen analyysi 1 tentti 17.3.2008

MAT-31106 Numerical Analysis 1 Exam 17.3.2008

Tentissä saa käyttää tavallista tai graafista/ohjemoitavaa laskinta ja yhtä kaksipuolista A4 sivua muistiinpanoja. Ei saa käyttää suurennessasia. Laskuissa välivaiheet on kirjottava näkyviin.

You are allowed to use a plain or graphing/programmable calculator and one two-sided A4 sheet of notes. A magnifying glass may not be used. Show all calculation steps.

- Arvioi suureen $\sqrt{x^2 + y^2}$ suhteellinen virhe ja anna oikeiden desimaalien lukumäärä, kun $x \approx 3.21$ (2 oikeaa desimaalia) ja $y \approx 7.654$ (3 oikeaa desimaalia).

Estimate the relative error and give the number of correct decimals of $\sqrt{x^2 + y^2}$ when $x \approx 3.21$ (2 correct decimals) and $y \approx 7.654$ (3 correct decimals).

Answer. Substituting $\bar{x} = 3.21$, $\bar{y} = 7.654$, $\Delta x = 0.005$ and $\Delta y = 0.0005$ into

$$\frac{\Delta(x^2 + y^2)^{1/2}}{(x^2 + y^2)^{1/2}} \approx \frac{\bar{x}\Delta x(\bar{x}^2 + \bar{y}^2)^{-1/2} + \bar{y}\Delta y(\bar{x}^2 + \bar{y}^2)^{-1/2}}{(\bar{x}^2 + \bar{y}^2)^{1/2}} = \frac{\bar{x}\Delta x + \bar{y}\Delta y}{\bar{x}^2 + \bar{y}^2}$$

gives relative error 2.9×10^{-4} . The absolute error is $\frac{\bar{x}\Delta x + \bar{y}\Delta y}{(\bar{x}^2 + \bar{y}^2)^{1/2}} = 0.0024 \leq 0.005$ and so the number of correct decimals is two.

- Selitä puolittamismenetelmän edut ja haitat Newtonin ja Raphsonin menetelmään verrattuna.

Discuss the advantages and disadvantages of the bisection method compared with the Newton-Raphson method.

Answer. Disadvantages: linear convergence rate (vs. quadratic for NR), need to provide *two* starting x values with different function sign (NR needs one starting value). Advantages: no need to compute derivative (NR requires derivatives), guaranteed convergence (NR can diverge), rigorous error bound (interval width) at every iteration (NR step size is not a rigorous bound on error).

- Etsi alla olevan taulukon Lagrangen interpolointipolynomi ja Newtonin interpolointipolynomi, ja todista (kirjoittamatta auki kumpakaan polynomia!) että ne ovat sama funktio.

x	0	1	3
$f(x)$	-0.0899	0.0100	0.1098

Find Lagrange's interpolating polynomial and Newton's interpolating polynomial for the above data, and prove (without expanding either of the polynomials!) that they are the same function.

Answer. Lagrange's form is

$$p_L(x) = (-0.0899) \frac{(x-1)(x-3)}{(0-1)(0-3)} + (0.0100) \frac{(x-0)(x-3)}{(1-0)(1-3)} + (0.1098) \frac{(x-0)(x-1)}{(3-0)(3-1)}$$

The divided difference table is

x	$f[\cdot]$	$f[\cdot, \cdot]$	$f[\cdot, \cdot, \cdot]$
0	-0.0899	0.0999	-0.016667
1	0.01	0.0499	
3	0.1098		

so Newton's form is

$$p_N(x) = -0.0899 + 0.0999x - 0.016667x(x-1)$$

The polynomials p_L and p_N are both of degree at most 2 and have the same values (as f) at the three distinct points $\{0, 1, 3\}$, so (by the theorem on uniqueness of interpolating polynomials) are the same function.

4. Laske $\int_1^3 x^x dx$ Rombergin menetelmää käyttäen (vähintään kolme saraketta).

Compute $\int_1^3 x^x dx$ using the Romberg method with at least three columns.

x	1	1.25	1.5	1.75	2	2.25	2.5	2.75	3
x^x	1	1.3217	1.8371	2.6627	4	6.2003	9.8821	16.1500	27

Answer: The first column is computed using recursive trapezoid rule:

$$\begin{aligned} T_1(2) &= \frac{2}{2}(1+27) = 28 \\ T_1(1) &= \frac{1}{2}T_1(2) + 1 \cdot (4) = 18 \\ T_1(\frac{1}{2}) &= \frac{1}{2}T_1(1) + \frac{1}{2}(1.8371 + 9.8821) = 14.8596 \\ T_1(\frac{1}{4}) &= \frac{1}{2}T_1(\frac{1}{2}) + \frac{1}{4}(1.3217 + 2.6627 + 6.2003 + 16.1500) = 14.0134 \end{aligned}$$

The second column is computed using extrapolation:

$$\begin{aligned} T_2(1) &= T_1(1) + \frac{T_1(1) - T_1(2)}{3} = 18 + \frac{18 - 28}{3} = 14.6667 \\ T_2(\frac{1}{2}) &= T_1(\frac{1}{2}) + \frac{T_1(\frac{1}{2}) - T_1(1)}{3} = 14.8596 + \frac{14.8596 - 18}{3} = 13.8128 \\ T_2(\frac{1}{4}) &= T_1(\frac{1}{4}) + \frac{T_1(\frac{1}{4}) - T_1(\frac{1}{2})}{3} = 14.0134 + \frac{14.0134 - 14.8596}{3} = 13.7313 \end{aligned}$$

The third and fourth columns:

$$\begin{aligned} T_3(\frac{1}{2}) &= 13.8128 + \frac{13.8128 - 14.6667}{15} = 13.7559 \\ T_3(\frac{1}{4}) &= 13.7313 + \frac{13.7313 - 13.8128}{15} = 13.7259 \\ T_4(\frac{1}{4}) &= 13.7259 + \frac{13.7259 - 13.7559}{63} = 13.7254 \end{aligned}$$

The Romberg table is

h	$T_1(h)$	$T_2(h)$	$T_3(h)$	$T_4(h)$
2	28			
1	18	14.6667		
0.5	14.8596	13.8128	13.7559	
0.25	14.0134	13.7313	13.7259	13.7254

and $\int_1^3 x^x dx \approx 13.7254$.

5. Etsi toisen asteen polynomi $p(x)$, joka on taulukossa olevan datan paras neliösumman likiarvo, ja kirjoita p ortogonaalisten polynomien summana.

x_i	-2	-1	1	2
$f(x_i)$	a	b	c	d

Find the degree-2 polynomial $p(x)$ that is the best least squares approximation of the data tabulated above, and express p as a sum of orthogonal polynomials.

Answer. The polynomials $p_0(x) = 1$ and $p_1(x) = x$ are orthogonal with respect to the inner product $(f, g) = f(-2)g(-2) + f(-1)g(-1) + f(1)g(1) + f(2)g(2)$. The next monic orthogonal polynomial is $p_2(x) = xp_1(x) - \gamma_1 p_0(x) = x^2 - \gamma_1$ with

$$\gamma_1 = \frac{(p_1, p_1)}{(p_0, p_0)} = \frac{10}{4} = \frac{5}{2}$$

The best least squares approximation is $p(x) = c_0 p_0(x) + c_1 p_1(x) + c_2 p_2(x) = c_0 + c_1 x + c_2(x^2 - \frac{5}{2})$ with

$$\begin{aligned} c_0 &= \frac{(f, p_0)}{(p_0, p_0)} = \frac{a+b+c+d}{4} \\ c_1 &= \frac{(f, p_1)}{(p_1, p_1)} = \frac{-2a-b+c+2d}{10} \\ c_2 &= \frac{(f, p_2)}{(p_2, p_2)} = \frac{1.5(a-b-c+d)}{9} = \frac{a-b-c+d}{6} \end{aligned}$$