TTY / Talous ja rakentaminen

Rakennustekniikka

RAK-32300 ELEMENTTIMENETELMÄN PERUSTEET, 4 op

Syksy 2018

1. Välikoe ratkaisut

to 04.10.2018

1. Ratkaise Galerkinin menetelmällä differentiaaliyhtälö

$$\frac{\mathbf{d^2u}}{\mathbf{dx^2}} + \mathbf{u} = \mathbf{2 \cdot x}, \qquad \mathbf{x} \in [\mathbf{0,1}], \qquad \mathbf{u(0) = u(1) = 0}$$

käyttäen kantafunktioita **$G_1(x)=x(x-1)$ ja $G_2(x)=x^2(x-1)$.**
Vertaa saatua tulosta tarkkaan ratkaisuun, kun **x=0,5**.

$$u(x)_{Exact} = 2\cdot x - \left(2\cdot \sin(x)\right) \ / \ \sin(1)$$

Basis functions $\quad G_1(x) = x\cdot(x-1)$ and $G_2(x) = x^2\cdot(x-1)$

$\Rightarrow$

$$\tilde{u}(x) = Q_1\cdot x\cdot(x-1) + Q_2\cdot x^2\cdot(x-1) = Q_1\cdot(x^2-x) + Q_2\cdot(x^3-x^2)$$

$$\tilde{u}'(x) = Q_1\cdot(2x-1) + Q_2\cdot(3x^2-2x)$$

$$\tilde{u}''(x) = 2Q_1 + Q_2\cdot(6x-2) = 2Q_1 + 2Q_2\cdot(3x-1)$$

$\Rightarrow$

$$L\tilde{u} - P = 2Q_1 + 2Q_2\cdot(3x-1) + Q_1\cdot(x^2-x) + Q_2\cdot(x^3-x^2) - 2\cdot x$$

$$\phi = \phi_1\cdot(x^2-x) + \phi_2\cdot(x^3-x^2)$$

$\Rightarrow$

$$\int_0^1 \phi\cdot(L\tilde{u}-P)\cdot dx = 0 \qquad \forall \phi_i$$

$\Rightarrow$ $\qquad\downarrow$ Valitse ; Choose $\downarrow$

$$\begin{cases} \int_0^1 (x^2-x)\cdot\left(2Q_1 + 2Q_2\cdot(3x-1) + Q_1\cdot(x^2-x) + Q_2\cdot(x^3-x^2) - 2\cdot x\right)\cdot dx = 0 & \leftarrow (\phi_1 = 1, \quad \phi_2 = 0) \\ \int_0^1 (x^3-x^2)\cdot\left(2Q_1 + 2Q_2\cdot(3x-1) + Q_1\cdot(x^2-x) + Q_2\cdot(x^3-x^2) - 2\cdot x\right)\cdot dx = 0 & \leftarrow (\phi_1 = 0, \quad \phi_2 = 1) \end{cases}$$

$\Rightarrow$

$$\begin{cases} \int_0^1 \begin{pmatrix} 2Q_1\cdot x^2 + 2Q_2\cdot(3x^3-x^2) + Q_1\cdot(x^4-x^3) + Q_2\cdot(x^5-x^4) - 2\cdot x^3 \ldots \\ -2Q_1\cdot x - 2Q_2\cdot(3x^2-x) - Q_1\cdot(x^3-x^2) - Q_2\cdot(x^4-x^3) + 2\cdot x^2 \end{pmatrix}\cdot dx = 0 \\ \int_0^1 \begin{pmatrix} 2Q_1\cdot x^3 + 2Q_2\cdot(3x^4-x^3) + Q_1\cdot(x^5-x^4) + Q_2\cdot(x^6-x^5) - 2\cdot x^4 \ldots \\ -2Q_1\cdot x^2 - 2Q_2\cdot(3x^3-x^2) - Q_1\cdot(x^4-x^3) - Q_2\cdot(x^5-x^4) + 2\cdot x^3 \end{pmatrix}\cdot dx = 0 \end{cases}$$

(... jatkuu ... cont'd)

$\Rightarrow$

$$\begin{cases} \displaystyle\int_0^1 \begin{pmatrix} 2Q_1\cdot x^2 + 2Q_2\cdot(3x^3 - x^2) + Q_1\cdot(x^4 - x^3) + Q_2\cdot(x^5 - x^4) - 2\cdot x^3 \ \ldots \\ -2Q_1\cdot x - 2Q_2\cdot(3x^2 - x) - Q_1\cdot(x^3 - x^2) - Q_2\cdot(x^4 - x^3) + 2\cdot x^2 \end{pmatrix}\cdot dx = 0 \\ \displaystyle\int_0^1 \begin{pmatrix} 2Q_1\cdot x^3 + 2Q_2\cdot(3x^4 - x^3) + Q_1\cdot(x^5 - x^4) + Q_2\cdot(x^6 - x^5) - 2\cdot x^4 \ \ldots \\ -2Q_1\cdot x^2 - 2Q_2\cdot(3x^3 - x^2) - Q_1\cdot(x^4 - x^3) - Q_2\cdot(x^5 - x^4) + 2\cdot x^3 \end{pmatrix}\cdot dx = 0 \end{cases}$$

(... jatkuu ... cont'd)

$\Rightarrow$

$$\begin{cases} \Big/_0^1 \begin{pmatrix} 2Q_1\cdot \frac{x^3}{3} + 2Q_2\cdot(3\cdot\frac{x^4}{4} - \frac{x^3}{3}) + Q_1\cdot(\frac{x^5}{5} - \frac{x^4}{4}) + Q_2\cdot(\frac{x^6}{6} - \frac{x^5}{5}) - 2\cdot\frac{x^4}{4} \ \ldots \\ -2Q_1\cdot\frac{x^2}{2} - 2Q_2\cdot(3\cdot\frac{x^3}{3} - \frac{x^2}{2}) - Q_1\cdot(\frac{x^4}{4} - \frac{x^3}{3}) - Q_2\cdot(\frac{x^5}{5} - \frac{x^4}{4}) + 2\cdot\frac{x^3}{3} \end{pmatrix} = 0 \\ \Big/_0^1 \begin{pmatrix} 2Q_1\cdot \frac{x^4}{4} + 2Q_2\cdot(3\cdot\frac{x^5}{5} - \frac{x^4}{4}) + Q_1\cdot(\frac{x^6}{6} - \frac{x^5}{5}) + Q_2\cdot(\frac{x^7}{7} - \frac{x^6}{6}) - 2\cdot\frac{x^5}{5} \ \ldots \\ -2Q_1\cdot\frac{x^3}{3} - 2Q_2\cdot(3\cdot\frac{x^4}{4} - \frac{x^3}{3}) - Q_1\cdot(\frac{x^5}{5} - \frac{x^4}{4}) - Q_2\cdot(\frac{x^6}{6} - \frac{x^5}{5}) + 2\cdot\frac{x^4}{4} \end{pmatrix} = 0 \end{cases}$$

(... jatkuu ... cont'd)

$\Rightarrow$

$$\begin{cases} \left(\frac{2}{3} - \frac{1}{20} - \frac{2}{2} + \frac{1}{12}\right)\cdot Q_1 + \left(\frac{5}{6} - \frac{1}{30} - \frac{6}{6} + \frac{1}{20}\right)\cdot Q_2 = -\frac{1}{6} \\ \left(\frac{2}{4} - \frac{1}{30} - \frac{2}{3} + \frac{1}{20}\right)\cdot Q_1 + \left(\frac{14}{20} - \frac{1}{42} - \frac{5}{6} + \frac{1}{30}\right)\cdot Q_2 = -\frac{1}{10} \end{cases} \Rightarrow \begin{bmatrix} -\frac{18}{60} & -\frac{9}{60} \\ -\frac{9}{60} & -\frac{39}{315} \end{bmatrix}\cdot\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \begin{Bmatrix} -\frac{1}{6} \\ -\frac{1}{10} \end{Bmatrix}$$

$\Rightarrow$

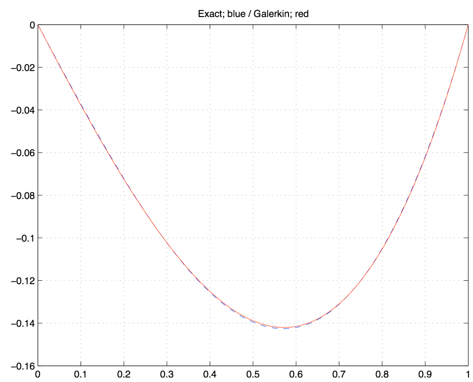
$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \begin{bmatrix} -\frac{18}{60} & -\frac{9}{60} \\ -\frac{9}{60} & -\frac{39}{315} \end{bmatrix}^{-1}\cdot\begin{Bmatrix} -\frac{1}{6} \\ -\frac{1}{10} \end{Bmatrix} = \frac{1}{\left(\frac{18\cdot 39}{60\cdot 315} - \frac{9\cdot 9}{3600}\right)}\cdot\begin{bmatrix} -\frac{39}{315} & \frac{9}{60} \\ \frac{9}{60} & -\frac{18}{60} \end{bmatrix}\cdot\begin{Bmatrix} -\frac{1}{6} \\ -\frac{1}{10} \end{Bmatrix} = \begin{Bmatrix} \frac{142}{369} \\ \frac{14}{41} \end{Bmatrix} \approx \begin{Bmatrix} 0{,}4011 \\ 0{,}3415 \end{Bmatrix}$$

$\Rightarrow$

Solution

$$\tilde{u}(x) = \frac{142}{369}\cdot(x^2 - x) + \frac{14}{41}\cdot\left(x^3 - x^2\right)$$

$$\tilde{u}(0{,}5) = \frac{142}{369}\cdot(0{,}5^2 - 0{,}5) + \frac{14}{41}\cdot\left(0{,}5^3 - 0{,}5^2\right) \approx -0{,}1389 \qquad \leftarrow\leftarrow (1)$$



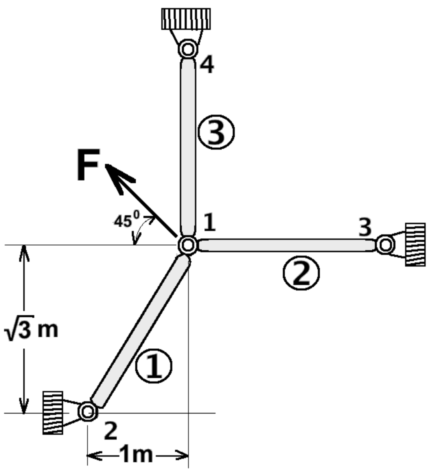
Exact solution

$$u(x)_{Exact} = 2\cdot x - 2\cdot \sin(x)/\sin(1)$$

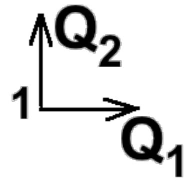
$$u(0{,}5)_{Exact} = 2\cdot 0{,}5 - 2\cdot\sin(0{,}5)/\sin(1) \approx -0{,}1395 \qquad \leftarrow\leftarrow (2)$$

$\Rightarrow \qquad (x = 0{,}5)$

The Galerkin approximate solution (1) $\cong$ The exact analytical solution of the differential equation (2)



2. Määritä kuvassa olevan kolmisauvaisen ristikon solmun **1** siirtymät ja sauvojen normaalijännitykset elementtimenetelmällä. Sauva **1** on pystysuoraan nähden kulmissa **30°**. Sauva **2** on vaakasuorassa. Sauva **3** on pystysuorassa. Sauvojen pituus on **2 m**. Materiaalin kimmomoduuli **E=70 GPa** ja sauvojen poikkipinta-ala **A=500 mm²** . Solmuun **1** vaikuttaa kuormitusvoima $\mathbf{F = 50000 \cdot \sqrt{2}\ N}$ vaakasuoraan nähden kulmassa **45°**. Laske lisäksi solmun **1** pystysuuntainen siirtymä, jos tämän solmun vaakasuuntainen siirtymä on estetty.



$$E = 70\ \text{GPa},\ A = 500 mm^2, \quad F = 50000 \cdot \sqrt{2}\, N, \quad L = 2\ m$$

Sauva 1 , Element 1

$$A_1 = 500 mm^2 = 500 \cdot 10^{-6} m^2, \quad 2 \to 1 \qquad l_1 = \frac{1}{2}, \quad m_1 = \frac{\sqrt{3}}{2}, \quad L_1 = 2\ m = L,$$

Sauva 2 , Element 2

$$A_2 = 500 mm^2 = 500 \cdot 10^{-6} m^2, \quad 3 \to 1 \qquad l_2 = -\frac{2}{2} = -1, \quad m_2 = \frac{0}{2} = 0, \quad L_2 = 2\ m = L,$$

Sauva 3 , Element 3

$$A_3 = 500 mm^2 = 500 \cdot 10^{-6} m^2, \quad 4 \to 1 \qquad l_3 = 0, \quad m_3 = -1, \quad L_3 = 2\ m = L,$$

Kaksi vapausastetta 2 DOF , (1 horizontal Q_1 , 2 vertical Q_2) , Node 1

$$k_e = \frac{EA}{l_e}\begin{bmatrix} l^2 & lm \\ lm & m^2 \end{bmatrix}, \quad k_1 = \frac{EA}{L}\begin{array}{cc} \begin{array}{cc} 1 & 2 \end{array} & \\ \begin{bmatrix} 1/4 & \sqrt{3}/4 \\ \sqrt{3}/4 & 3/4 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array}, \quad k_2 = \frac{EA}{L}\begin{array}{cc} \begin{array}{cc} 1 & 2 \end{array} & \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array}, \quad k_3 = \frac{EA}{L}\begin{array}{cc} \begin{array}{cc} 1 & 2 \end{array} & \\ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array}$$

$\Rightarrow$

$$[K] = \sum_{i=1}^{3} k_i = \frac{EA}{L}\begin{array}{cc} \begin{array}{cc} 1 & 2 \end{array} & \\ \begin{bmatrix} 5/4 & \sqrt{3}/4 \\ \sqrt{3}/4 & 7/4 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array} \qquad \{F\} = \begin{Bmatrix} -50000 \\ 50000 \end{Bmatrix}\begin{array}{c} 1 \\ 2 \end{array}$$

$\Rightarrow$

$$[K]\{Q\} = \{F\} \qquad \det = 5 \cdot 7/16 - \left(\sqrt{3}/4\right)^2 = \frac{32}{16}$$

$\Rightarrow$

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = [K]^{-1}\{F\} = \frac{L}{EA} \cdot \frac{1}{\det}\begin{bmatrix} 7/4 & -\sqrt{3}/4 \\ -\sqrt{3}/4 & 5/4 \end{bmatrix}\begin{Bmatrix} -50000 \\ 50000 \end{Bmatrix} =$$

$\rightarrow$

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \frac{2000 \cdot 16}{70 \cdot 10^3 \cdot 500 \cdot 32}\begin{Bmatrix} (7/4) \cdot (-50000) - (\sqrt{3}/4) \cdot 50000 \\ (\sqrt{3}/4) \cdot 50000 + (5/4) \cdot 50000 \end{Bmatrix} \approx \begin{Bmatrix} -3{,}1186 \\ 2{,}4043 \end{Bmatrix} \text{mm}$$

Stress in the element 1

Node 2 fixed $\Rightarrow$ $q_1 = q_2 = 0$, $q_3 = Q_1$, $q_4 = Q_2$ *Node* 2 $\rightarrow$ Node 1

$$\sigma_1 = \frac{E}{L_1}\begin{bmatrix}-l_1 & -m_1 & l_1 & m_1\end{bmatrix}\begin{Bmatrix}q_1\\q_2\\q_3\\q_4\end{Bmatrix} = \frac{70\cdot 10^3}{2000}\left[\frac{1}{2}\cdot\left(-3{,}1186\right)+\frac{\sqrt{3}}{2}\cdot(2{,}4043)\right]\approx 18{,}3 \text{ MPa}$$

$\Rightarrow$

$\sigma_1 = 18$ MPa

- -

Stress in the element 2

Node 3 fixed $\Rightarrow$ $q_1 = q_2 = 0$, $q_3 = Q_1$, $q_4 = Q_2$ *Node* 3 $\rightarrow$ Node 1

$$\sigma_2 = \frac{E}{L_2}\begin{bmatrix}-l_2 & -m_2 & l_2 & m_2\end{bmatrix}\begin{Bmatrix}q_1\\q_2\\q_3\\q_4\end{Bmatrix} = \frac{70\cdot 10^3}{2000}\left[(-1)\cdot\left(-3{,}1186\right)+(0)\cdot(2{,}4043)\right]\approx 109{,}1 \text{ MPa}$$

$\Rightarrow$

$\sigma_2 = 109$ MPa

- -

Stress in the element 3

Node 4 fixed $\Rightarrow$ $q_1 = q_2 = 0$, $q_3 = Q_1$, $q_4 = Q_2$ *Node* 4 $\rightarrow$ Node 1

$$\sigma_3 = \frac{E}{L_3}\begin{bmatrix}-l_3 & -m_3 & l_3 & m_3\end{bmatrix}\begin{Bmatrix}q_1\\q_2\\q_3\\q_4\end{Bmatrix} = \frac{70\cdot 10^3}{2000}\left[0\cdot\left(-3{,}1186\right)-1\cdot(2{,}4043)\right]\approx \text{-}84{,}2 \text{ MPa}$$

$\Rightarrow$

$\sigma_3 = -84$ MPa

Vain pystysuuntainen liike solmussa 1; Only vertical displacement of the node 1 ; Q_3

Vapausaste Q_1 eliminoidaan ; Elimination of Q_1

$$[K] = \sum_{i=1}^{3} k_i = \frac{EA}{L} \begin{array}{cc} & 2 \\ & \\ \left[\begin{matrix} 5/4 & \sqrt{3}/4 \\ \sqrt{3}/4 & 7/4 \end{matrix}\right] & 2 \end{array} \qquad \{F\} = \left\{\begin{matrix} -50000 \\ 50000 \end{matrix}\right\} 2$$

$\Rightarrow$

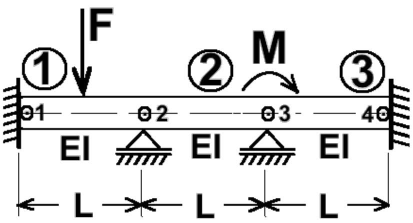
$$\frac{E \cdot A}{L} \cdot \frac{7}{4} \cdot Q_3 = 50000$$

$\Rightarrow$

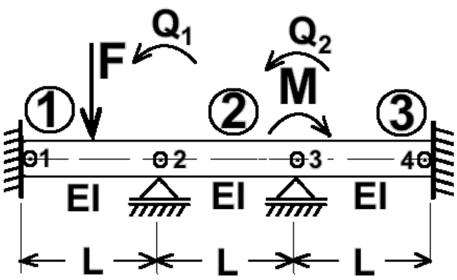
Siirtymä : Displacement Q_3

$$Q_3 = \frac{L}{E \cdot A} \cdot \frac{4}{7} \cdot 50000 = \frac{2000}{70000 \cdot 500} \cdot \frac{4}{7} \cdot 50000 \approx 1{,}63 \text{ mm} \quad \leftarrow\leftarrow$$

3. Määritä elementtimenetelmällä kuvan päistään päistään jäykästi ja keskeltä solmuista **2** ja **3** vapaasti tuetun vaakapalkin kiertymät, kun elementin **1** keskellä vaikuttaa pistevoima **F** alaspäin ja solmussa **3** vaikuttaa pistemomentti **M**. Määritä lisäksi elementin **2** keskipisteen taipuma ja taivutusmomentti vasemmassa päässä solmun **1** kohdalla. Palkin taivutusjäykkyys on **EI**. Palkin pituus on **3L**. Palkki on venymätön. Käytä ratkaisussa kolmea elementtiä.



E = 200 GPa, I = 10^{-4} m^4, L = 2 m, F = 100 kN, M = 100 kNm



$$\left[k_1\right] = \frac{EI}{L^3} \begin{array}{c} \begin{array}{cccc} & & & 1 \end{array} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & \mathbf{4L^2} \end{bmatrix} \begin{array}{c} \\ \\ \\ 1 \end{array} \end{array} , \quad \left[k_2\right] = \frac{EI}{L^3} \begin{array}{c} \begin{array}{cccc} & 1 & & 2 \end{array} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & \mathbf{4L^2} & -6L & \mathbf{2L^2} \\ -12 & -6L & 12 & -6L \\ 6L & \mathbf{2L^2} & -6L & \mathbf{4L^2} \end{bmatrix} \begin{array}{c} \\ 1 \\ \\ 2 \end{array} \end{array}$$

$$\left[k_3\right] = \frac{EI}{L^3} \begin{array}{c} \begin{array}{cccc} & 2 & & \end{array} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & \mathbf{4L^2} & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{array}{c} \\ 2 \\ \\ \\ \end{array} \end{array}$$

$\Rightarrow$

After elimination

$$\left[\mathrm{K}\right] = \frac{EI}{L} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 8 & 2 \\ 2 & 8 \end{bmatrix} \begin{array}{c} 1 \\ 2 \end{array} \end{array} , \quad \det[\,] = 8\cdot 8 - (2)\cdot(2) = 60$$

Loading vector Equivalent nodal moment (node 2) and point moment (node 3) (Nodes 2 and 3)

$$\left\{\mathrm{F}\right\} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{Bmatrix} \dfrac{\mathrm{F}\cdot \mathrm{L}}{8} \\ -M \end{Bmatrix} = \begin{Bmatrix} 25000 \\ -100000 \end{Bmatrix} \cdot 10^3 \text{ Nmm}$$

$\Rightarrow$

$\Rightarrow$

Kiertymät Q_1 ja Q_2 Slopes Q_1 and Q_2

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = [K]^{-1} \cdot \{F\} = \frac{L}{EI} \cdot \frac{1}{\det} \cdot \begin{bmatrix} 8 & -2 \\ -2 & 8 \end{bmatrix} \cdot \begin{Bmatrix} \dfrac{F \cdot L}{8} \\ -M \end{Bmatrix} = \frac{2000}{2 \cdot 10^5 \cdot 10^8} \cdot \frac{1}{60} \cdot \begin{bmatrix} 8 & -2 \\ -2 & 8 \end{bmatrix} \cdot \begin{Bmatrix} \dfrac{100000 \cdot 2}{8} \\ -100000 \end{Bmatrix} \cdot 10^3$$

$\Rightarrow$

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \frac{2000}{2 \cdot 10^5 \cdot 10^8} \cdot \frac{1}{60} \cdot \begin{Bmatrix} 8 \cdot \dfrac{100000 \cdot 2}{8} - 2 \cdot 100000 \\ -2 \cdot \dfrac{100000 \cdot 2}{8} + 8 \cdot (-100000) \end{Bmatrix} \cdot 10^3 \approx \begin{Bmatrix} 6{,}667 \cdot 10^{-4} \\ -1{,}40 \cdot 10^{-3} \end{Bmatrix} \qquad \leftarrow$$

$\Rightarrow$

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} \approx \begin{Bmatrix} 0{,}03820^\circ \\ -0{,}0812^\circ \end{Bmatrix}$$

Taipuma vasemmanpuoleisten tukien keskellä

The deflection at the midpoint between the left supports (-0,375 mm)

Element 1

$$H_1 = \frac{1}{4}(2 - 3\xi + \xi^3) \quad , \quad H_2 = \frac{1}{4}(1 - \xi - \xi^2 + \xi^3)$$

$$H_3 = \frac{1}{4}(2 + 3\xi - \xi^3) \quad , \quad H_4 = \frac{1}{4}(-1 - \xi + \xi^2 + \xi^3)$$

$$v(\xi) = H_1 q_1 + \frac{l_e}{2} H_2 q_2 + H_3 q_3 + \frac{l_e}{2} H_4 q_4$$

$$q_1 = q_2 = q_3 = 0 \quad , \quad q_4 = Q_1$$

$\Rightarrow$

$$v(0) = \frac{L}{2} \cdot H_4(0) \cdot Q_1 - \frac{F \cdot L^3}{192 \cdot E \cdot I} = -2000 \cdot \frac{6{,}667 \cdot 10^{-4}}{2 \cdot 4} - 0{,}2083 \approx -0{,}375 \text{ mm} \qquad \leftarrow$$

$\Rightarrow$

$$v(0) \approx -0{,}375 \text{ mm} \qquad (\text{"Exact" } -0{,}375 \text{ mm}) \qquad \leftarrow$$

Taipuma keskimmäisten tukien keskellä
The deflection at the midpoint of the two mid supports

Element 2

$$H_1 = \frac{1}{4}(2 - 3\xi + \xi^3) \quad , \quad H_2 = \frac{1}{4}(1 - \xi - \xi^2 + \xi^3)$$

$$H_3 = \frac{1}{4}(2 + 3\xi - \xi^3) \quad , \quad H_4 = \frac{1}{4}(-1 - \xi + \xi^2 + \xi^3)$$

$$v(\xi) = H_1 q_1 + \frac{l_e}{2} H_2 q_2 + H_3 q_3 + \frac{l_e}{2} H_4 q_4$$

$$q_1 = q_3 = 0 \quad , \quad q_2 = Q_1 \quad , \quad q_4 = Q_2$$

$\Rightarrow$

$$v(0) = \frac{L}{2} \cdot H_2(0) \cdot Q_1 + \frac{L}{2} \cdot H_4(0) \cdot Q_2 = \frac{2000}{2 \cdot 4} \cdot \left(6{,}667 \cdot 10^{-4} + 1{,}4 \cdot 10^{-3}\right) \approx 0{,}517 \text{ mm}$$

$\Rightarrow$

$$v(0) \approx 0{,}517 \text{ mm} \qquad \left(\text{"Exact"} \;\; 0{,}521 \text{ mm}\right) \qquad \leftarrow$$

Taipuma oikeanpuoleisten tukien keskellä
The deflection at the midpoint of the element 3

Element 3

$$H_1 = \frac{1}{4}(2 - 3\xi + \xi^3) \quad , \quad H_2 = \frac{1}{4}(1 - \xi - \xi^2 + \xi^3)$$

$$H_3 = \frac{1}{4}(2 + 3\xi - \xi^3) \quad , \quad H_4 = \frac{1}{4}(-1 - \xi + \xi^2 + \xi^3)$$

$$v(\xi) = H_1 q_1 + \frac{l_e}{2} H_2 q_2 + H_3 q_3 + \frac{l_e}{2} H_4 q_4$$

$$q_1 = q_3 = q_3 = 0 \quad , \quad q_2 = Q_2$$

$\Rightarrow$

$$v(0) = \frac{L}{2} \cdot H_2(0) \cdot Q_2 = \frac{2000}{2 \cdot 4} \cdot \left(-1{,}4 \cdot 10^{-3}\right) \approx -0{,}35 \text{ mm}$$

$\Rightarrow$

$$v(0) \approx -0{,}35 \text{ mm} \qquad \left(\text{"Exact"} \;\; \text{-}\,0{,}354 \text{ mm}\right) \qquad \leftarrow$$

Taivutusmomentti vasemmassa päässä ; The bending moment of the left fixed support

Element 1 ; Node 1

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix} = [k_1] \cdot \{Q\} - \begin{Bmatrix} -F/2 \\ -F \cdot L/8 \\ -F/2 \\ F \cdot L/8 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ \mathbf{6L} & \mathbf{4L^2} & \mathbf{-6L} & \mathbf{2L^2} \\ -12 & -6L & 12 & -6L \\ 6L & 2\mathrm{L}^2 & -6L & 4L^2 \end{bmatrix} \cdot \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \mathbf{Q_1} \end{Bmatrix} - \begin{Bmatrix} -F/2 \\ -F \cdot L/8 \\ -F/2 \\ F \cdot L/8 \end{Bmatrix}$$

$\Rightarrow$

$$M_1 = -R_2 = -(\frac{EI}{L^3} \cdot \left(2L^2 \cdot Q_1\right) + \frac{F \cdot L}{8}) = -(\frac{2 \cdot EI}{L} \cdot Q_1 + \frac{F \cdot L}{8}) = -\left(13334000 + 25000 \cdot 10^3\right)$$

$\Rightarrow$

$M_1 = 38{,}33$ kNm $\leftarrow\leftarrow$

Choose basis functions $\mathbf{G_i}$. Determine the coefficients $\mathbf{Q_i}$ in $\tilde{\mathbf{u}} = \sum_{i=1}^{n} \mathbf{Q_i G}_i$, such that

$\int_V \phi(L\tilde{u} - P)\, dV = 0$ for every ϕ of the type $\phi = \sum_{i=1}^{n} \phi_i G_i$.

$$k_e = \frac{EA_e}{l_e}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix} \quad , \quad l = \frac{x_2 - x_1}{l_e} \quad , \quad m = \frac{y_2 - y_1}{l_e}$$

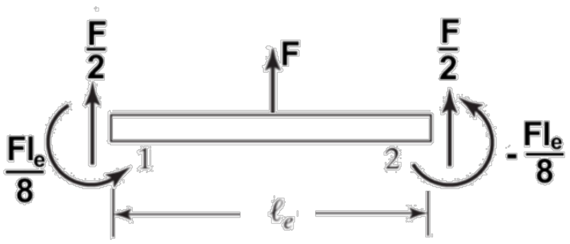
$$\sigma = \frac{E_e}{l_e}\begin{bmatrix} -l & -m & l & m \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

$$k_e = \frac{EI}{l_e^3}\begin{bmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{bmatrix}$$

$$H_1 = \frac{1}{4}(2 - 3\xi + \xi^3) \quad , \quad H_2 = \frac{1}{4}(1 - \xi - \xi^2 + \xi^3)$$

$$H_3 = \frac{1}{4}(2 + 3\xi - \xi^3) \quad , \quad H_4 = \frac{1}{4}(-1 - \xi + \xi^2 + \xi^3)$$

$$v(\xi) = H_1 q_1 + \frac{l_e}{2} H_2 q_2 + H_3 q_3 + \frac{l_e}{2} H_4 q_4$$



$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \cdot \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} - \begin{Bmatrix} F/2 \\ FL/8 \\ F/2 \\ -FL/8 \end{Bmatrix}$$